Master's Thesis

Coherent Sheaves on Non-Archimedean Analytic Spaces

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Introduction

Modern non-archimedean analytic geometry started with the introduction of rigid analytic spaces by John Tate, who unfortunately passed away very recently. His notes circulated since 1962, although this work was only published officially nine years later in [Tat71]. Tate's rigid analytic geometry tries to mimic complex analytic geometry over complete non-archimedean fields like the p-adic numbers $\mathbb{Q}_p$. The basic building blocks of this theory are the Tate algebras $k\{T_1, \dots, T_n\}$ of power series converging on the n-dimensional closed unit disc, as well as their quotient Banach algebras, which were called affinoid algebras by Tate. The category of affinoid spaces is defined to be the opposite category of the category of affinoid algebras.

In Tate's approach, the set of points of an affinoid space $X = \mathrm{Sp}(A)$ is the set of maximal ideals of the affinoid algebra A. Functions $f \in A$ can be evaluated at points $x \in X$ and as the valuation of k extends uniquely to the finite extension field $A / \mathfrak{m}_x$, the values of functions at points have defined absolute values. The space $X = \mathrm{Sp}(A)$ carries a Grothendieck topology whose admissible subsets are subsets which are cut out by non-strict inequalities like $\{ x \in X \,|\, |f(x)| \leq 1 \}$. Tate defined a sheaf of functions on X, which allows to globalize the theory and to consider global rigid analytic spaces which look locally like affinoid spaces.

Rigid analytic spaces admit a well-behaved theory of coherent sheaves, similar to the case of locally Noetherian schemes, which was first developed by Reinhardt Kiehl in [Kie67b]. For example, the category of coherent sheaves on an affinoid space $X = \mathrm{Sp}(A)$ is equivalent to the category of finitely generated A-modules.

In the 90's, several variations on the theory of rigid analytic spaces were suggested by Vladimir Berkovich in [Ber90] and subsequent work. Most importantly, he suggested to take more points into account than just the maximal ideals of an affinoid algebra A; namely certain multiplicative seminorms on A. Equivalently, instead of characters on A with values in finite field extensions of k, he allows characters with values in arbitrary complete non-archimedean field extensions of k. In order to underline the difference from rigid analytic geometry, we will follow Berkovich in the notation and write $X = \mathcal{M}(A)$ instead of $X = \mathrm{Sp}(A)$ for these spaces. The benefit is now that besides the Grothendieck topology introduced by Tate, the set $X = \mathcal{M}(A)$ carries also an ordinary point-set topology, in which e.g. $\{ x \in X \mid |f(x)| < 1 \}$ is an open set, as one would expect. Analytic spaces with the Berkovich topology have very good point-set topological properties, good singular cohomology groups, interesting homotopy types and are much closer to actual complex analytic spaces.

The second novelty introduced by Berkovich was to generalize the notion of affinoid algebras from quotients of the Tate algebras $k\{T_1, \dots, T_n\}$ to quotients of the generalized Tate algebras $k\{r_1^{-1}T_1, \dots, r_n^{-1}T_n\}$ of power series converging on the n-dimensional polydisc of polyradius $(r_1, \dots, r_n)$. This algebra is strictly affinoid in the sense of Tate only if the r_i belong to $\sqrt{|k^\times|}$. Berkovich's alteration therefore makes it possible to let the radius of discs vary continuously over the real numbers. This allows, among other things, to also have a reasonable analytic geometry over arbitrary fields equipped with trivial valuation.

Also after the appearance of [Ber93], the foundations of the theory continued to undergo some slight changes. For example, in [Ber93], Berkovich changed his definition of analytic spaces to what has now become the standard by allowing more pathological spaces leading however to a more flexible category. The analytic spaces from [Ber90] can be recovered as the so-called good analytic spaces. Regarding the notion of proper morphisms, the original definition in rigid analytic geometry by Kiehl was somewhat ad hoc and basic properties like stability under composition were not known to be true for some time. In full generality, these issues were solved by Michael Temkin in [Tem00; Tem04], where a more satisfying definition of proper morphisms between Berkovich analytic spaces was given and where their basic properties as well as the equivalence to Kiehl's original definition (in the strict case) were proven. Temkin also clarified a number of other foundational questions, showing for example that the category of strictly analytic spaces from [Ber93] is a full subcategory of the category of arbitrary analytic spaces. A polished account of the modern usage of terminology related to Berkovich analytic spaces is given in the expository article [Tem15].

In this thesis, I want to give self-contained proofs of the results of Kiehl on coherent sheaves in the setting of Berkovich analytic spaces. I decided to adopt the modern conventions, thus bringing concepts and arguments from different periods of time, using different languages, together in one unified account. For example, some effort has to go into generalizing results on strictly affinoid algebras over fields with non-trivial valuation, considered by Tate and the school of rigid analytic geometry, to the more general affinoid algebras introduced by Berkovich. The usual way to do this is to perform scalar extension along a certain extension $k \hookrightarrow k_r$ of the ground field. To my knowledge, no detailed account of these arguments exists in the literature.

In Chapter 1, I recall some basics of the theory of analytic spaces in the sense of Berkovich, or more precisely of his approach from [Ber93]. We will discuss the different topologies on analytic spaces and show in particular that for sheaf-theoretic questions there is no difference between the analytic G-topology with analytic domains as admissible subsets, and the weak G-topology with affinoid domains as admissible subsets. We will also briefly discuss some important properties of analytic spaces and their morphisms like compactness, separatedness and properness and describe the procedure of reducing questions on affinoid spaces to the strict case using a ground field extension $k \hookrightarrow k_r$.

In Chapter 2, we will develop the general theory of coherent sheaves on non-archimedean analytic spaces, which behaves quite similarly to the theory of coherent sheaves on locally Noetherian schemes. In particular, we shall prove Kiehl's theorem which establishes an equivalence between the category of coherent sheaves on an affinoid space $X = \mathcal{M}(A)$ and the category of finite A-modules. In algebraic geometry one always has an equivalence between quasi-coherent sheaves on $\mathrm{Spec}(A)$ and A-modules, which for Noetherian A specializes to an equivalence between coherent sheaves and finite A-modules. In complex analytic geometry, it is already a deep theorem, due to Oka, that the structure sheaf of a complex analytic space is coherent. The non-archimedean analytic situation is somewhere in between: As in algebraic geometry, there is a fully faithful, exact functor associating to an A-module M a sheaf $\widetilde{M}$ on $\mathcal{M}(A)$, and it is easy to see that for finite M, the associated sheaf $\widetilde{M}$ will be coherent, due to the fact that A is a Noetherian ring. In particular, $\mathcal{O}_X$ is a coherent sheaf if X is an analytic space. The converse however, i.e. the fact that any coherent sheaf on an affinoid space is associated to a finite A-module is less trivial than in algebraic geometry, and it is not true that the sheaves associated to arbitrary A-modules can be characterized as the quasi-coherent ones.

With respect to the theory of coherent sheaves, Noetherian schemes compared to locally Noetherian schemes behave like compact analytic spaces compared to general analytic spaces. For example, we will establish the fact that the category of coherent sheaves on a compact analytic space is a Noetherian abelian category and we will formulate the Artin-Rees lemma for coherent sheaves on a compact analytic space.

Chapter 3 is devoted to a proof of a non-archimedean version of Grauert's proper mapping theorem from complex analytic geometry, asserting the coherence of the higher direct images of a coherent sheaf under a proper morphism of analytic spaces. The statement is easily reduced to the case of an affinoid base space, i.e. a proper morphism $\varphi : X \to \mathcal{M}(B)$. We will first show in this situation that if $\mathcal{F}$ is a coherent sheaf on X, then the cohomology groups $H^q(X, \mathcal{F})$ are finite modules over the algebra B (so in particular, coherent sheaves on a proper analytic space have finite-dimensional cohomology over k). As a next step, we are going to prove a version of the theorem on formal functions, i.e. we will show that if $b \in B$, then $\lim_i H^q(X, \mathcal{F}/b^i\mathcal{F})$ is isomorphic to the b-adic completion of $H^q(X, \mathcal{F})$. This will help us to deduce that the higher direct image sheaf $R^q\varphi_*(\mathcal{F})$ is associated to the finite B-module $H^q(X, \mathcal{F})$ and, in particular, coherent.

It is perhaps interesting to remark that the algebraic geometry version of Grauert's theorem on higher direct images of coherent sheaves under a proper morphism $\varphi : X \to Y$, where X and Y are finite type schemes over a field k, can be deduced from the analytic version which we are going to prove in this thesis. This can be done by equipping the ground field k with the trivial valuation and using a GAGA translation between finite type k-schemes and k-analytic spaces. Some details on this approach are given in [Duc15]. This nicely illustrates some of the benefits of the greater generality of Berkovich's theory compared to rigid analytic geometry.

1 Preliminaries

Berkovich analytic spaces come equipped both with an ordinary point-set topology and a so-called G-topology. The construction of the analytic G-topology on Berkovich analytic spaces depends on the notions of quasinets and nets, which we are going to describe in Section 1.2. In general, given a net on a topological space, one can construct an associated G-topology. On k-affinoid spaces, or more generally, k-analytic spaces, the analytic G-topology will be defined as the G-topology associated to the net of affinoid domains. We will show that the analytic G-topology gives rise to the same sheaf topos as the weak G-topology given by affinoid domains, so that for many sheaf-theoretic questions, we can work with the simpler weak G-topology.

It is still necessary to have the analytic G-topology, because morphisms of analytic spaces will, in general, not be continuous with respect to the weak G-topology, but only with respect to the analytic G-topology.

In Section 1.5, we are going to describe the scalar extension functor of analytic spaces along an extension of complete non-archimedean fields $k \hookrightarrow l$. This will be important to us, because for $l = k_r$, this provides a technique for reducing questions about general analytic spaces to strictly analytic spaces over non-trivially valued fields.

In the remaining sections, we are going to describe some important properties of analytic spaces and their morphisms. In particular, we are going to introduce proper morphisms, which will be our main objects of study in Chapter 3.

1.1 G-topological spaces

We define G-topological spaces as in [BGR84, Def. 9.1.1/1]:

Definition 1.1.1. Let X be a set. A *G-topology* on X is given by certain distinguished subsets $U \subseteq X$, called *admissible open subsets*, as well as, for each admissible open subset $U \subseteq X$, a system of certain distinguished set-theoretic coverings $U = \bigcup_{i \in I} U_i$ by other admissible open subsets $U_i \subseteq X$, called *admissible coverings*, in such a way that the following axioms are satisfied:

(i) The intersection $U \cap V$ of two admissible open subsets $U, V \subseteq X$ is again admissible open.

(ii) For each admissible open subset $U \subseteq X$, the trivial covering $\{ U \}$ of U is admissible.

(iii) If U is an admissible open subset of X, $(U_i)_{i \in I}$ is an admissible covering of U, and for each $i \in I$, the family $(V_{ij})_{j \in J_i}$ is an admissible covering of U_i, then the family $(V_{ij})_{i \in I, j \in J_i}$ is an admissible covering of U.

(iv) If $U, V \subseteq X$ are two admissible open subsets of X and $V \subseteq U$, and if $(U_i)_{i \in I}$ is an admissible covering of U, then the family $(V \cap U_i)_{i \in I}$ is an admissible covering of V.

The set X, together with a fixed G-topology, is called a *G-topological* space.

A G-topology on a set X gives rise in an obvious way to a Grothendieck topology in the sense of [Art62], so that we may, in particular, speak about *sheaves* on a G-topological space.

It is sometimes convenient to have a slightly more general notion than G-topological spaces available. For a *generalized G-topology* (this terminology is ad hoc and not standard), we replace axiom (i) with the requirement that the intersection of two admissible open subsets $U, V \subseteq X$ be admissible open, *whenever U and V are contained in a common third admissible open subset W*. This corresponds to the observation, that for general Grothendieck topologies, the underlying category of a Grothendieck topology need not admit all binary products, but has to admit fiber products. If the category admits a terminal object—or in the context of G-topologies, if all of X is admissible open—then there are automatically all finite products, resp. intersections.

A G-topology can satisfy the following additional completeness properties:

Definition 1.1.2. Let X be a G-topological space. We call X *saturated* if the following additional properties are satisfied:

(G1) The subsets $\emptyset, X \subseteq X$ are admissible open.

(G2) Let $U \subseteq X$ be an admissible open subset, $(U_i)_{i\in I}$ an admissible covering of U and let $V \subseteq U$ be any subset. If every intersection $V \cap U_i$, $i \in I$ is admissible open, then V is admissible open.

(G3) Let $(U_i)_{i\in I}$ be a set-theoretic covering of an admissible open subset $U \subseteq X$. If $(U_i)_{i\in I}$ admits a refinement which is admissible, then $(U_i)_{i\in I}$ itself is an admissible covering.

Note that actual topological spaces are always saturated. For actual topologies and more generally for saturated G-topologies, the following condition is trivial. It ensures that two G-topologies give rise to the same notion of sheaves.

Definition 1.1.3. Let X be a set and let T, T' be two (generalized) G-topologies on X. The G-topology T' is called *slightly finer* than T if the following is satisfied:

(i) The G-topology T' is *finer* than T, i.e. every T-open subset is T'-open and every T-admissible covering of a T-open subset by T-open subsets is T'-admissible.

(ii) The G-topology T is a *basis* of T', i.e. every T'-open subset admits a T'-admissible covering by T-open subsets.

(iii) Every T'-admissible covering of a T-open subset by T-open subsets admits a refinement to a T-admissible covering by T-open subsets.

In the situation of Definition 1.1.3, every T-sheaf extends uniquely to a sheaf on T', and the same holds for morphisms of sheaves. In more precise terms:

Proposition 1.1.4. *Let X be a set and let T, T' be two generalized G-topologies on X. Assume that T' is slightly finer than T. Then the functor*

$$\mathrm{Sh}(X, T') \to \mathrm{Sh}(X, T)$$

restricting sheaves on (X, T') to sheaves on (X, T) is an equivalence of topoi.

Proof. See [BGR84, Prop. 9.2.3/1] for a proof for G-topologies. The same proof works for generalized G-topologies as well. □

A *ringed (generalized) G-topological space* is a pair $(X, \mathcal{O}_X)$, consisting of a (generalized) G-topological space X and a sheaf of rings $\mathcal{O}_X$ on X. One can then speak about sheaves of $\mathcal{O}_X$-modules. The above proposition carries over to sheaves of modules:

Proposition 1.1.5. *Let X be a set, and let T, T' be two (generalized) G-topologies on X. Assume that T' is slightly finer than T. Let $\mathcal{O}_T$ be a sheaf of rings on (X, T) and let $\mathcal{O}_{T'}$ be its extension to (X, T'). Then the functor*

$$\mathrm{Mod}(\mathcal{O}_{T'}) \to \mathrm{Mod}(\mathcal{O}_T)$$

restricting a sheaf of modules on $(X, T', \mathcal{O}_{T'})$ to a sheaf of modules on $(X, T, \mathcal{O}_T)$ is an equivalence of (abelian, monoidal, ...) categories.

1.2 Quasinets and nets

Quasinets will play the role of admissible coverings in the analytic G-topology on non-archimedean analytic spaces.

Definition 1.2.1. Let X be a topological space, $U \subseteq X$ a subset and $\mathfrak{U} = (U_i)_{i \in I}$ a family of subsets of U. The family $\mathfrak{U}$ is called a *quasinet* on U, if the following holds: For every $x \in U$, there are finitely many indices $i_1, \dots, i_n \in I$ such that $U_{i_1} \cup \dots \cup U_{i_n}$ is a neighbourhood of x in U, and such that $x \in U_{i_1} \cap \dots \cap U_{i_n}$ holds.

Note that a quasinet is, in particular, a set-theoretic covering.

Remark 1.2.2. If the sets U_i are all closed in U, then the requirement that $x \in U_{i_1} \cap \dots \cap U_{i_n}$ in the definition of a quasinet is not necessary. Indeed, if say $x \notin U_{i_n}$, then $U \setminus U_{i_n}$ is a neighbourhood of x in U and since $(U_{i_1} \cap \dots \cap U_{i_n}) \cap (U \setminus U_{i_n}) \subseteq U_{i_1} \cup \dots \cup U_{i_{n-1}}$, so is $U_{i_1} \cup \dots \cup U_{i_{n-1}}$. In this way, one can successively eliminate all U_{i_k} which do not contain x.

In particular, every finite covering $U = U_1 \cup \cdots \cup U_n$ is a quasinet on U, provided that the U_i are closed in U. For example, this is automatically satisfied if U is Hausdorff and the U_i are compact.

Lemma 1.2.3. *Let X be a topological space and $U \subseteq X$ a compact subset. Then every quasinet $\mathfrak{U}$ on U admits a finite subcovering (and by Remark 1.2.2, this subcovering is again a quasinet if the covering sets are closed in U).*

Proof. By assumption, every point $x \in U$ has a neighbourhood of the form $U_{i_1} \cup \cdots \cup U_{i_n}$ with $i_1, \ldots, i_n \in I$. Therefore, the topological interiors of these finite unions form an open covering of U. As U is compact, there exists a finite subcovering of this open covering, and in particular, U is covered by finitely many U_i. □

Quasinets have the following stability properties which are easy to verify:

Lemma 1.2.4. *Let X be a topological space.*

(i) *Let $U, V \subseteq X$ be two subsets of X and assume that $V \subseteq U$. Let $(U_i)_{i \in I}$ be a quasinet on U. Then $(V \cap U_i)_{i \in I}$ is a quasinet on V.*
(ii) *Let $U \subseteq X$ be a subset, $(U_i)_{i \in I}$ a quasinet on U and suppose that for each $i \in I$, we are given a quasinet $(V_{ij})_{j \in J_i}$ on U_i. Then the family $(V_{ij})_{i \in I, j \in J_i}$ is a quasinet on U.*
(iii) *Let $U \subseteq X$ be a subset of X, $\mathfrak{U}$ a set-theoretic covering of U and let $\mathfrak{V}$ be a refinement of $\mathfrak{U}$, which is a quasinet on U. Then $\mathfrak{U}$ is a quasinet on U as well.*

Definition 1.2.5. Let X be a topological space and τ a family of subsets of X. The family τ is called a *net* if it is a quasinet on X and for each $U, V \in \tau$, the family $\tau|_{U \cap V} := \{ W \in \tau \mid W \subseteq U \cap V \}$ is a quasinet on $U \cap V$.

A net τ on a topological space X gives rise to a G-topology T on X which we call the G-topology *associated to the net* τ. A subset $U \subseteq X$ is admissible open in T if there is a quasinet $(U_i)_{i \in I}$ on U consisting of subsets $U_i \in \tau$. A covering $U = \bigcup_{i \in I} U_i$ is admissible if the family $(U_i)_{i \in I}$ is a quasinet on U.

Proposition 1.2.6. *The above definitions indeed determine a G-topology on X which is in fact saturated.*

Proof. We show that the intersection of two admissible open subsets is again admissible open, since here the assumption that τ is a net is used. All other properties follow immediately from Lemma 1.2.4.

So assume that $U, V \subseteq X$ are admissible open. Let $(U_i)_{i\in I}$, resp. $(V_j)_{j\in J}$ be quasinets on U, resp. V with $U_i, V_i \in \tau$. By Lemma 1.2.4 (i), $(U \cap V_j)_{j\in J}$ is a quasinet on $U \cap V$. By the same statement, for each $j \in J$, the family $(U_i \cap V_j)_{i\in I}$ is a quasinet on $U \cap V_j$. From Lemma 1.2.4 (ii) it follows that $(U_i \cap V_j)_{i\in I, j\in J}$ is a quasinet on $U \cap V$. As τ is a net, each of the subsets $U_i \cap V_j$ is covered by a quasinet $U_i \cap V_j = \bigcup_{k\in K_{ij}} W_{ijk}$ with $W_{ijk} \in \tau$. Again by Lemma 1.2.4 (ii), the W_{ijk} form a quasinet on $U \cap V$. This proves the claim. □

A net τ on a topological space X is called *dense* if for every $U \in \tau$ and every $x \in U$ there exists a neighbourhood basis of x in U consisting of sets $V \in \tau|_U$.

Proposition 1.2.7. *Let X be a topological space and τ a dense net on X. Then the G-topology associated to τ is finer than the original topology on X.*

Proof. Let us first show that every open subset $\mathcal{U}$ of X is admissible open in the G-topology associated to τ. For this we have to show that $\tau|_{\mathcal{U}} = \{ U \in \tau \mid U \subseteq \mathcal{U} \}$ is a quasinet on $\mathcal{U}$. So let $x \in \mathcal{U}$ be arbitrary. Since τ is in particular a quasinet on X, there are finitely many sets $V_1, \dots, V_n \in \tau$ with $x \in V_1 \cap \dots \cap V_n$ such that $V_1 \cup \dots \cup V_n$ is a neighbourhood of x in X. Then for each $i \in [1 \mathrel{..} n]$, $\mathcal{U} \cap V_i$ is an open neighbourhood of x in V_i. By denseness of τ, there exists a $W_i \in \tau$ such that $W_i \subseteq \mathcal{U} \cap V_i$ and such that W_i is a neighbourhood of x in $\mathcal{U} \cap V_i$. Then $W_1 \cup \dots \cup W_n$ is a neighbourhood of x in $(V_1 \cup \dots \cup V_n) \cap \mathcal{U}$, and hence in $\mathcal{U}$.

That every open covering $\mathcal{U} = \bigcup_{i\in I} \mathcal{U}_i$ is a quasinet is clear. □

1.3 Affinoid spaces

From now on, let k be a complete non-archimedean field. As described in [Ber90, Sec. 1.2 and Sec. 2.2], a k-affinoid spectrum $X = \mathcal{M}(A)$, where A is a k-affinoid algebra, carries both the ordinary Berkovich topology, making it a compact Hausdorff space, and a G-topology whose admissible open subsets are the affinoid subdomains of X and whose admissible coverings are the finite ones. We call this G-topology the *weak G-topology* on X. If we restrict ourselves to strictly affinoid algebras and strictly affinoid subdomains, then this strict weak G-topology corresponds precisely to the classical weak G-topology on the rigid space $\mathrm{Sp}(A)$ (by which we mean that the associated sites are isomorphic).

Since finite intersections of affinoid subdomains are again affinoid subdomains, it is clear that they form a net on the topological space $X = \mathcal{M}(A)$ with the Berkovich topology. We call the associated G-topology the *analytic G-topology* on X. The admissible open subsets of the analytic G-topology are called *analytic domains* of X. Concretely, this means that a subset $U \subseteq X$ is an analytic domain in X if there exists a quasinet $(U_i)_{i\in I}$ on U, consisting of affinoid subdomains of X. A covering $U = \bigcup_{i\in I} U_i$ is admissible in the analytic G-topology if it is a quasinet.

Proposition 1.3.1. *Let $X = \mathcal{M}(A)$ be a k-affinoid spectrum. Then the analytic G-topology on X is slightly finer than the weak G-topology.*

Proof. Affinoid domains are always compact and, in particular, closed in X. From Remark 1.2.2 it follows that the analytic G-topology is finer than the weak G-topology. The affinoid domains form a basis of the analytic G-topology by its construction. The last condition of Definition 1.1.3 is satisfied by Lemma 1.2.3. □

It follows that the sheaf $V \mapsto A_V$ from [Ber90, Prop. 2.2.5] extends uniquely to the analytic G-topology. We shall write $\mathcal{O}_X$ for this extension, so that, in particular, we have $\mathcal{O}_X(V) = A_V$ if V is an affinoid domain in $X = \mathcal{M}(A)$.

Lemma 1.3.2. *If the field k is non-trivially valued then the net of affinoid domains on a k-affinoid spectrum $X = \mathcal{M}(A)$ is dense.*

Proof. This follows from [Ber90, Prop. 2.2.3 (iii)]. □

In the case of a non-trivially valued field k, Proposition 1.2.7 therefore implies that the analytic G-topology on X is finer than the Berkovich topology. That is to say that for an open subset $\mathcal{U} \subseteq X$, the family $\tau|_{\mathcal{U}} = \{ V \subseteq \mathcal{U} \mid V \text{ is an affinoid domain in } X \}$ is a quasinet on $\mathcal{U}$. Let $\mathcal{O}_{X_B}$ be the restriction of the sheaf $\mathcal{O}_X$ to the Berkovich topology. For open $\mathcal{U} \subseteq X$ the sheaf condition implies

$$\mathcal{O}_{X_B}(\mathcal{U}) = \mathcal{O}_X(\mathcal{U}) = \varprojlim_{V \in \tau|_{\mathcal{U}}} \mathcal{O}_X(U).$$

This limit is not directed; if one wishes for a directed limit, one has to take the limit over all compact analytic domains (a.k.a. special domains) instead of only the affinoid ones. This yields exactly the description from [Ber90, Sec. 2.3] of the sheaf $\mathcal{O}_{X_B}$.

1.4 Analytic spaces

Let X be a locally Hausdorff topological space. A *k-affinoid atlas* on X consists of a net τ on X together with a functor $A_{(-)} : (\tau, \subseteq)^{\mathrm{op}} \to \{ k\text{-affinoid algebras} \}$, which sends an inclusion $V \subseteq U$ of subsets $U, V \in \tau$ to a morphism $A_U \to A_V$ corresponding to an embedding of affinoid domains and, finally, for each $U \in \tau$ a homeomorphism of topological spaces $i_U : U \xrightarrow{\sim} \mathcal{M}(A_U)$ such that for $V \subseteq U$ the diagram

$$\begin{tikzcd} V \arrow[r] \arrow[d, "i_V"'] & U \arrow[d, "i_U"] \\ \mathcal{M}(A_V) \arrow[r] & \mathcal{M}(A_U) \end{tikzcd}$$

always commutes. One can then readily identify U as a topological space with $\mathcal{M}(A_U)$, as well as in the case $U \subseteq V$ with an affinoid subdomain in V (or rather in $\mathcal{M}(A_V)$).

By [Ber93, Prop. 1.2.13], there exists a finest k-affinoid atlas refining τ. This finest atlas satisfies the following saturatedness property: Namely that whenever $U \in \tau$ holds and $V \subseteq U$ is an affinoid domain in U, then V belongs to τ as well; this follows from [Ber93, Prop. 1.2.13 (ii)]. The elements of this finest k-affinoid atlas are called *affinoid subdomains* of the k-analytic space X. The G-topology associated to the net of affinoid subdomains is called the *analytic G-topology* on X and its admissible open subsets are called *analytic domains* of X. In contrast to the affinoid case, the affinoid domains don't necessarily form a G-topology in the strict sense of Definition 1.1.1. However they still form a *generalized* G-topology. Indeed, if $U, V \subseteq X$ are affinoid domains which are contained in a common third affinoid domain $W \subseteq X$, then $U \cap V$ is again an affinoid domain in W which, by the saturatedness of the k-affinoid atlas of all k-affinoid domains, is also a k-affinoid domain in X. We call the generalized G-topology of all affinoid domains and finite coverings the *weak generalized G-topology* on the k-analytic space X.

If A is a k-affinoid algebra, and $X = \mathcal{M}(A)$, then one can start with the k-affinoid atlas $(X, \tau, A_{(-)}) = (X, \{X\}, X \mapsto A)$ on X. The affinoid domains in the corresponding k-analytic space are precisely the affinoid domains of the k-affinoid spectrum $\mathcal{M}(A)$ in the original sense, and also A_V for a k-affinoid domain $V \subseteq X$ coincides with its original meaning. The k-analytic spaces arising in this way will be called *k-affinoid spaces*.

Proposition 1.4.1. *Let X be a k-analytic space. Then the analytic G-topology on X is slightly finer than the weak generalized G-topology.*

Proof. The proof proceeds as in Proposition 1.3.1 except for that we cannot conclude that affinoid domains are closed in X, as X need not necessarily be Hausdorff. Affinoid domains are, however, still closed in other affinoid domains which is all we need. $\square$

Let X be a k-analytic space. The association $V \mapsto A_V$ of affinoid subdomains on X is a sheaf in the weak generalized G-topology of X. This follows immediately from the corresponding fact in the affinoid case. It follows that this sheaf extends uniquely to a sheaf in the analytic G-topology which we shall denote by $\mathcal{O}_X$.

Lemma 1.4.2. *If the field k is non-trivially valued then the net of affinoid domains on a k-analytic space X is dense.*

Proof. This follows from [Ber90, Prop. 2.2.3 (iii)]. □

In case of a non-trivially valued ground field, it follows that the sheaf $\mathcal{O}_X$ restricts to a sheaf in the Berkovich topology which we shall denote by $\mathcal{O}_{X_B}$.

A morphism of k-analytic spaces $\varphi\colon X \to Y$ is a map of the underlying sets which is continuous both with respect to the respective Berkovich topologies and with respect to the respective analytic G-topologies, together with a morphism of sheaves $\mathcal{O}_Y \to \varphi_*(\mathcal{O}_X)$ such that for every affinoid domain $V \subseteq Y$ and every affinoid domain $U \subseteq \varphi^{-1}(V)$ in X, the morphism $\mathcal{O}_X(V) \to \mathcal{O}_X(\varphi^{-1}(V)) \to \mathcal{O}_X(U)$ is a morphism of k-affinoid algebras (i.e. bounded with respect to the respective Banach norms). This is not Berkovich's original definition given in [Ber93, after Prop. 1.2.10], but can be shown to be equivalent to it (see [Tem15, Sec. 4.1.2]). We choose this definition, because it is easier to handle theoretically. In order to actually construct morphisms of analytic spaces given by concrete atlases, the characterization of [Ber93] is more suitable.

Proposition 1.4.3. *The functor*

$$\{\, k\text{-affinoid algebras} \,\}^{\mathrm{op}} \to \{\, k\text{-analytic spaces} \,\}, \qquad A \mapsto \mathcal{M}(A)$$

sending a k-affinoid algebra to its associated k-affinoid space is fully faithful.

1.5 Scalar extension

Let $k \hookrightarrow l$ be an extension of complete non-archimedean fields. If $X = \mathcal{M}(A)$ is a k-affinoid space, then we call the l-affinoid space $l \mathbin{\hat{\otimes}_k} X := \mathcal{M}(l \mathbin{\hat{\otimes}_k} A)$ the *scalar extension* of X along $k \hookrightarrow l$, where $l \mathbin{\hat{\otimes}_k} A$ denotes the completed tensor product of Banach algebras. This construction gives rise to a functor

$$\{\, k\text{-affinoid spaces} \,\} \to \{\, l\text{-affinoid spaces} \,\}, \qquad X \mapsto l \mathbin{\hat{\otimes}_k} X.$$

If $U \subseteq X$ is an affinoid domain, then so is $l \mathbin{\hat{\otimes}}_k U \hookrightarrow l \mathbin{\hat{\otimes}}_k X$. In appropriate categories (e.g. topological spaces or G-topological spaces or "analytic spaces over k"), there is a morphism $\pi : l \mathbin{\hat{\otimes}}_k X \to X$, and in these categories one always has $l \mathbin{\hat{\otimes}}_k U = \pi^{-1}(U)$ in an appropriate sense.

If $\mathfrak{U} = (U_i)_{i \in I}$ is an admissible affinoid covering of an affinoid domain $U \subseteq X$, then $l \mathbin{\hat{\otimes}}_k \mathfrak{U} \coloneqq (l \mathbin{\hat{\otimes}}_k U_i)_{i \in I}$ is an admissible affinoid covering of $l \mathbin{\hat{\otimes}}_k U$.

These constructions can be generalized to k-analytic spaces, i.e. there is a functor

$$\{\, k\text{-analytic spaces} \,\} \to \{\, l\text{-analytic spaces} \,\}, \qquad X \mapsto l \mathbin{\hat{\otimes}}_k X,$$

which, for affinoid X, restricts to the functor described above. If $U \subseteq X$ is an analytic domain in X, then $l \mathbin{\hat{\otimes}}_k U$ is an analytic domain in $l \mathbin{\hat{\otimes}}_k X$, and it is the preimage of U under the map $l \mathbin{\hat{\otimes}}_k X \to X$ in appropriate categories. If $\mathfrak{U}$ is an admissible covering of an analytic domain $U \subseteq X$, then we denote by $l \mathbin{\hat{\otimes}}_k \mathfrak{U}$ the associated covering of $l \mathbin{\hat{\otimes}}_k U$ in $l \mathbin{\hat{\otimes}}_k X$.

For details on the construction of the scalar extension, we refer the reader to [Ber93, Sec. 1.4].

Scalar extension, combined with appropriate descent arguments, can often be used to reduce questions to the case of non-trivial valuation and strictly analytic spaces. In order to do this, one considers the extension of complete non-archimedean fields $k \hookrightarrow k_r$ where $r = (r_1, \ldots, r_s) \in \mathbb{R}_{>0}^s$ is a finite tuple of real numbers which are $\mathbb{Q}$-linearly independent in the (multiplicative) $\mathbb{Q}$-vector space $\mathbb{R}_{>0} / \sqrt{|k^\times|}$. The field k_r is introduced in [Ber90, Sec. 2.1]. Given finitely many k-affinoid spaces U_i, $i \in [1 \,..\, n]$, there is always a tuple r such that all of the k_r-affinoid spaces $k_r \mathbin{\hat{\otimes}}_k U_i$, $i \in [1 \,..\, n]$ are strictly affinoid.

Remark 1.5.1. It seems to be appropriate to remark here that Berkovich in [Ber93] describes a more refined theory than we use here. Namely, he introduces a category of Φ_k-analytic spaces, where Φ_k is a class of k-affinoid spaces satisfying certain properties. Examples for such classes are the class of all k-affinoid spaces (which we use here) or the class of strictly k-affinoid spaces, or the class of k_H-affinoid spaces where H is a subgroup of $\mathbb{R}_{>0}$, as used by Temkin in [Tem15]. In particular, there is a notion of strictly k-analytic spaces (for non-trivially valued k). There is an obvious forgetful functor from the category of strictly k-analytic spaces to the category of arbitrary k-analytic spaces. This functor is fully faithful, so that it is legitimate to view strictly k-analytic spaces as k-analytic spaces of a particular kind, but this is a non-trivial theorem due to Temkin [Tem04, Cor. 4.10].

If X is a compact separated k-analytic space, then an extension $k \hookrightarrow k_r$ can be chosen in such a way that $k_r \hat{\otimes}_k X$ is a strictly k_r-analytic space in the sense that it is contained in the essential image of the forgetful functor from strictly k_r-analytic spaces to arbitrary k_r-analytic spaces.

1.6 Separated morphisms

Similarly to the scalar extension along a field extension, one can construct fiber products in the category of k-analytic spaces. In the case that $X = \mathcal{M}(B)$, $Y = \mathcal{M}(C)$, $Z = \mathcal{M}(A)$, the fiber product is given by $X \times_Z Y = \mathcal{M}(B \hat{\otimes}_A C)$. For details we again refer to [Ber93, Sec. 1.4].

The fiber product of a morphism $\varphi\colon X \to Y$ along an embedding of an analytic domain $V \hookrightarrow Y$ is given by the preimage $\varphi^{-1}(V)$.

A morphism $\varphi\colon X \to Y$ of k-analytic spaces is called a *closed immersion*, if for every affinoid domain $V \subseteq Y$, the preimage $\varphi^{-1}(V)$ is an affinoid domain, and the morphism of k-affinoid algebras $\mathcal{O}_Y(V) \to \mathcal{O}_X(\varphi^{-1}(V))$ is admissible and surjective.

A morphism $\varphi\colon X \to Y$ of k-analytic spaces is called *separated* if the diagonal morphism $\Delta\colon X \to X \times_Y X$ is a closed immersion. Obviously, every morphism between k-affinoid spaces is separated. A k-analytic space is called *separated* if the morphism $X \to \mathcal{M}(k)$ is separated. The basic properties of separated morphisms can be established as in algebraic geometry:

Proposition 1.6.1. (i) *If $\varphi\colon X \to Y$ and $\psi\colon Y \to Z$ are separated morphisms then $\psi \circ \varphi$ is also separated.*

(ii) *If*

$$\begin{array}{ccc} X' & \xrightarrow{\varphi'} & Y' \\ \downarrow & & \downarrow \\ X & \xrightarrow{\varphi} & Y \end{array}$$

is a cartesian diagram of k-analytic spaces and φ is separated, then so is φ'.

(iii) *Let $\varphi\colon X \to Y$ be a morphism of k-analytic spaces and let $Y = \bigcup_{i\in I} Y_i$ be an admissible covering of Y by k-analytic domains. Then φ is separated if and only if $\varphi\colon \varphi^{-1}(Y_i) \to Y_i$ is separated for every $i \in I$.*

(iv) *If $\varphi\colon X \to Y$ is a separated morphism of k-analytic spaces and $k \hookrightarrow l$ is an extension of complete non-archimedean fields then $l \mathbin{\hat{\otimes}_k} \varphi\colon l \mathbin{\hat{\otimes}_k} X \to l \mathbin{\hat{\otimes}_k} Y$ is also separated.*

In particular, if $\varphi\colon X \to Y$ is a separated morphism and Y is a separated space (for example, because it is affinoid), then X is also separated. As in algebraic geometry, one can show:

Proposition 1.6.2. *If X is a separated k-analytic space and $U, V \subseteq X$ are affinoid subdomains of X, then the intersection $U \cap V$ is again an affinoid subdomain.*

In other words, the weak generalized G-topology on a separated k-analytic space is in fact an actual G-topology in the strict sense of Definition 1.1.1.

1.7 Proper morphisms

Let B be a k-affinoid algebra and let $\Phi\colon A \to D$ be a morphism of Banach B-algebras from a k-affinoid Banach B-algebra A to a Banach B-algebra D. This homomorphism is called *inner with respect to B* if there exists an $r \in \mathbb{R}^n_{>0}$ and an admissible surjective homomorphism

$$\Psi\colon B\{r_1^{-1}T_1, \dots, r_n^{-1}T_n\} \to A, \qquad T_i \mapsto f_i,$$

such that $\rho(\Phi(f_i)) < r_i$ for all $i \in [1 \,..\, n]$. Here ρ denotes the spectral radius of an element of some Banach algebra.

Lemma 1.7.1. *Assume that k is non-trivially valued, let B be a strictly k-affinoid algebra and let $\Phi\colon A \to D$ be a morphism of Banach B-algebras where A is strictly k-affinoid. If Φ is inner with respect to B, then it is* strictly *inner, meaning that there exists an admissible surjective homomorphism*

$$\Psi\colon B\{T_1, \dots, T_n\} \to A, \qquad T_i \mapsto f_i$$

such that $\rho(\Phi(f_i)) < 1$ for all $i \in [1 \,..\, n]$.

Proof. Choose $\Psi\colon B\{r_1^{-1}T_1, \dots, r_n^{-1}T_n\} \to A$, $T_i \mapsto f_i$ such that $\rho(\Phi(f_i)) < r_i$. We first show that we may assume that $r_i \in \sqrt{|k^\times|}$. Note that we have $\rho(\Phi(f_i)) \le \rho(f_i) \le r_i$ and that $\rho(f_i) \in \sqrt{|k^\times|} \cup \{0\}$ by [Ber90, Cor. 2.1.6], because A is strictly k-affinoid. If r_i is not an element of $\sqrt{|k^\times|}$, we can therefore choose $s_i \in \sqrt{|k^\times|}$ such that $\rho(f_i) \le s_i < r_i$. Replacing r_i with s_i does not effect the surjectivity of Ψ (and admissibility is then automatic over a non-trivially valued field), so we can assume that $r_i \in \sqrt{|k^\times|}$.

Inspecting the proofs of [BGR84, Thm. 6.1.5/4, Prop. 6.1.1/5], we find that there is an admissible surjective homomorphism $B\{T_1', \dots, T_m'\} \to A$ such that each T_i' is mapped to an element of the form $a \cdot f_j^\nu$ with $a \in k$, $j \in [1 \,..\, n]$, $\nu \in \mathbb{N}_{>0}$ and $|a| \cdot r_j^\nu = 1$. These elements then satisfy $\rho(\Phi(a \cdot f_j^\nu)) < |a| \cdot r_j^\nu \le 1$. □

Let $\varphi\colon U = \mathcal{M}(A) \to Y = \mathcal{M}(B)$ be a morphism of k-affinoid spaces. We say that a point $x \in U$ is an *interior point* of U with respect to Y if the morphism $A \to \mathcal{H}(x)$ is inner with respect to B. Note that the inequality $\rho(\Phi(f_i)) < r_i$ then reads $|f_i(x)| < r_i$. We write $\mathrm{Int}(U/Y)$ for the set of interior points of U with respect to Y.

Again, let $\varphi\colon U = \mathcal{M}(A) \to Y = \mathcal{M}(B)$ be a morphism of k-affinoid spaces. We say that an affinoid subdomain $V \subseteq U$ lies *interior to* U with respect to Y if the restriction morphism $A \to \mathcal{O}_U(V)$ is inner with respect to B. The inequality $\rho(\Phi(f_i)) < r_i$ is then equivalent to $|f_i(x)| < r_i$ for all $x \in V$. We use the notation $V \Subset_Y U$ to express that V is interior in U with respect to Y.

Lemma 1.7.2. *Let $\varphi\colon U = \mathcal{M}(A) \to Y = \mathcal{M}(B)$ be a morphism of k-affinoid spaces and let $x \in U$. Then we have $x \in \operatorname{Int}(U/Y)$ if and only if there exists an affinoid domain $V \subseteq Y$ such that $x \in V$ and $V \Subset_Y U$. The affinoid domain V can always be chosen in such a way that it is a neighbourhood of x in U. If k is non-trivially valued and U, Y are strictly k-affinoid, then V can be chosen as a strictly affinoid subdomain of U.*

Proof. It is clear that $x \in V \Subset_Y U$ implies $x \in \operatorname{Int}(U/Y)$. Conversely suppose that $x \in \operatorname{Int}(U/Y)$ and let $\Phi\colon B\{r_1^{-1}T_1, \dots, r_n^{-1}T_n\} \to A$, $T_i \mapsto f_i$ be such that $|f_i(x)| < r_i$ for all $i \in [1 \,..\, n]$. Choose $s_i \in \mathbb{R}_{>0}$ such that $|f_i(x)| < s_i < r_i$. In the strict case, we may assume that $s_i \in \sqrt{|k^\times|}$. Then the Weierstraß domain

$$V \coloneqq U\{s_1^{-1}f_1, \dots, s_n^{-1}f_n\} = \{\, y \in U \mid |f_i(y)| \le s_i \,\}$$

satisfies $V \Subset_Y U$ and is a neighbourhood of x in U. □

Remark 1.7.3. Assume that k is non-trivially valued and let $\varphi\colon U = \mathcal{M}(A) \to Y = \mathcal{M}(B)$ be a morphism of strictly k-affinoid spaces. In [BGR84, Sec. 9.6.2], a strictly affinoid domain V satisfying $V \Subset_Y U$ is called *relatively compact* in U, with respect to Y. The equivalence of their definition of relative compactness with our definition of relatively interior domains in the strict situation follows from Lemma 1.7.1.

Lemma 1.7.4. *Let $Y = \mathcal{M}(B)$ be a k-affinoid space.*

(i) *Let $\varphi\colon U \to Y$ be a morphism of k-affinoid spaces and let $V \Subset_Y U$ be a relatively interior affinoid subdomain of U. If $\varphi\colon Y' = \mathcal{M}(B') \to Y = \mathcal{M}(B)$ is a morphism of k-affinoid spaces then $V \times_Y Y' \Subset_{Y'} U \times_Y Y'$.*

(ii) *Let $\varphi_1\colon U_1 \to Y$ and $\varphi_2\colon U_2 \to Y$ be morphisms of k-affinoid domains and let $V_1 \Subset_Y U_1$ and $V_2 \Subset_Y U_2$ be relatively interior domains. Then $V_1 \times_Y V_2 \Subset_y U_1 \times_Y U_2$.*

(iii) *Let $\varphi\colon X \to Y$ be a separated morphism of k-analytic spaces and let $U_1, U_2, V_1, V_2 \subseteq X$ be affinoid subdomains such that $V_1 \Subset_Y U_1$ and $V_2 \Subset_Y U_2$. Then also $V_1 \cap V_2 \Subset_Y U_1 \cap U_2$.*

Proof. The proof works as in [BGR84, Lem. 9.6.2/1]. □

Now let $\varphi\colon X \to Y$ be a morphism of k-analytic spaces. A point $x \in X$ is called an *interior point of X with respect to Y* if for every affinoid subdomain $W \subseteq Y$ containing $\varphi(x)$, there exists an affinoid domain $U \subseteq \varphi^{-1}(W)$ which is a neighbourhood of x in $\varphi^{-1}(W)$ and such that $x \in \operatorname{Int}(U/W)$. We write $\operatorname{Int}(X/Y)$ for the set of all interior points of X with respect to Y. The morphism φ is called *boundaryless* if $\operatorname{Int}(X/Y) = X$.

The morphism $\varphi\colon X \to Y$ is called *proper* if it is separated, boundaryless and compact, where the latter means that preimages of compact analytic domains are again compact.

Proposition 1.7.5. (i) *If $\varphi\colon X \to Y$ and $\psi\colon Y \to Z$ are proper morphisms then $\psi \circ \varphi$ is also proper.*

(ii) *If*

$$\begin{tikzcd} X' \arrow[r, "\varphi'"] \arrow[d] & Y' \arrow[d] \\ X \arrow[r, "\varphi"] & Y \end{tikzcd}$$

is a cartesian diagram of k-analytic spaces and φ is proper, then so is φ'.

(iii) *Let $\varphi\colon X \to Y$ be a morphism of k-analytic spaces and let $Y = \bigcup_{i\in I} Y_i$ be an admissible covering of Y by k-analytic domains. Then φ is proper if and only if $\varphi\colon \varphi^{-1}(Y_i) \to Y_i$ is proper for every $i \in I$.*

(iv) *If $\varphi\colon X \to Y$ is a proper morphism of k-analytic spaces and $k \hookrightarrow l$ is an extension of complete non-archimedean fields, then $l \mathbin{\hat{\otimes}_k} \varphi\colon l \mathbin{\hat{\otimes}_k} X \to l \mathbin{\hat{\otimes}_k} Y$ is also proper.*

Proof. The statements about base change are not difficult to show. The statement regarding compositions follows from [Tem04, Cor. 5.7]. The locality statement is shown in [Tem04, Cor. 5.6]. □

We will make use of the following property of proper morphisms with affinoid base:

Proposition 1.7.6. *Let $\varphi\colon X \to Y = \mathcal{M}(B)$ be a proper morphism with affinoid base. Then X is compact and separated and there exist two finite* *admissible affinoid coverings* $\mathfrak{U} = (U_i)_{i\in I}$, $\mathfrak{V} = (V_i)_{i\in I}$ *such that* $V_i \Subset_Y U_i$ *for every* $i \in I$.

Proof. The space X is compact and separated because Y and φ are compact and separated. Let $x \in X$ be an arbitrary point. Then as φ is boundaryless, there exists an affinoid domain $U_x \subseteq X$ which is a neighbourhood of $x \in X$ and such that $x \in \mathrm{Int}(U/Y)$. By Lemma 1.7.2, there exists an affinoid domain $V_x \subseteq U_x$ such that $V_x \Subset_Y U_x$ and such that V_x is a neighbourhood of x in U_x and hence in X. Since X is compact, there is a finite subcovering of the covering $(V_x)_{x\in X}$. □

Remark 1.7.7. Let us say that a morphism $\varphi\colon X \to Y = \mathcal{M}(B)$ of (strictly) k-analytic spaces with affinoid base satisfies condition (†) if it is separated and if there exist two finite affinoid coverings $(U_i)_{i\in I}$, $(V_i)_{i\in I}$ such that $V_i \Subset_Y U_i$ for all $i \in I$. Kiehl in [Kie67a] calls a morphism of strictly analytic (or rather rigid analytic) spaces $\varphi\colon X \to Y$ proper if there exists an admissible affinoid covering $Y = \bigcup_{j\in J} Y_j$ such that for each $j \in J$, the morphism $\varphi\colon \varphi^{-1}(Y_j) \to Y_j$ satisfies condition (†). This definition is also used in the encyclopedic work [BGR84].

Note that with this definition it is not obvious that a proper morphism with affinoid base satisfies condition (†), i.e. that this condition is local with respect to affinoid subdomains.

From Proposition 1.7.6 it follows that properness in our sense implies properness in the sense of Kiehl. The fact that Kiehl's notion of properness for strictly k-analytic spaces is actually equivalent to the one we chose is due to Temkin [Tem00, Cor. 4.5]. In the same work, the basic properties from Proposition 1.7.5 were established in the strict case. For strictly analytic spaces over a discretely valued field, these results had been obtained earlier by Werner Lütkebohmert using methods of formal geometry [Lüt90]. In [Tem04] the results were generalized to the not necessarily strict case using graded reduction techniques.

2 Coherent sheaves on analytic spaces

In this chapter, we are going to establish the general properties of coherent sheaves on k-analytic spaces where k is a complete non-archimedean field. Since coherence is a local property, it is fundamental to understand coherent sheaves on affinoid spaces. It will turn out that there is an equivalence between the category of coherent sheaves on an affinoid space $X = \mathcal{M}(A)$ and the category of finite A-modules, a result which was first shown by Kiehl in the language of rigid analytic geometry [Kie67b].

In Section 2.1, we are going to describe the coherent sheaf on $\mathcal{M}(A)$ associated to a finite A-module M. This construction is quite similar to the construction of associated sheaves in algebraic geometry.

We are then going to collect some basic facts about coherent sheaves on arbitrary G-topological spaces in Section 2.2. Applied to analytic spaces, this will imply that a sheaf $\mathcal{F}$ on an analytic space X is coherent if and only if it is $\mathfrak{U}$-coherent for some admissible affinoid covering $\mathfrak{U}$ of X, which means that for each $U \in \mathfrak{U}$ the restriction of $\mathcal{F}$ to U is associated to a finite module.

In Section 2.4 we will describe the scalar extension of coherent sheaves along an extension of complete non-archimedean fields, in order to be able to reduce a proof of Kiehl's theorem to the strictly affinoid case.

Section 2.5 is devoted to the proof of Kiehl's classification of coherent sheaves on affinoid spaces. Combined with Tate's acyclicity theorem, Kiehl's theorem implies that coherent sheaves on affinoid spaces have vanishing Čech cohomology. We will however prove this directly as an intermediate result towards the proof of Kiehl's theorem.

The cohomological triviality of coherent sheaves on affinoid spaces implies as in algebraic geometry that the sheaf cohomology of a coherent sheaf on a separated analytic space can be computed as the Čech cohomology with respect to an affinoid covering. We will capture these results in Section 2.6

In Section 2.7, we will establish some basic results on coherent sheaves on compact analytic spaces, which behave somewhat analogous to coherent sheaves on Noetherian schemes.

2.1 Associated sheaves

Let $X = \mathcal{M}(A)$ be a k-affinoid space and let M be an A-module. Setting $\widetilde{M}(U) \coloneqq \mathcal{O}_X(U) \otimes_A M$ for an affinoid domain $U \subseteq X$ defines a presheaf $\widetilde{M}$ of $\mathcal{O}_X$-modules in the weak G-topology of X.

If M is a finite A-module then by [Ber90, Prop. 2.1.9], M carries a unique norm up to equivalence which provides M with the structure of a finite Banach A-module. Then the finite $\mathcal{O}_X(U)$-modules $\widetilde{M}(U)$ also carry in a unique way the structure of finite Banach $\mathcal{O}_X(U)$-modules and there is a canonical isomorphism $\widetilde{M}(U) \cong \mathcal{O}_X(U) \hat{\otimes}_A M$ in the category of Banach modules and bounded linear maps.

Theorem 2.1.1 (Tate acyclicity). *Let $X = \mathcal{M}(A)$ be a k-affinoid space and let M be an A-module. The presheaf $\widetilde{M}$ is Čech acyclic with respect to every finite admissible covering $\mathfrak{U} = (U_i)_{i \in I}$ of X, i.e. the augmented Čech complex*

$$0 \to \widetilde{M}(X) \to \prod_{i \in I} \widetilde{M}(U_i) \to \prod_{i,j \in I} \widetilde{M}(U_i \cap U_j) \to \dots$$

is exact.

If M is a finite module then the Čech complex is admissible and exact, i.e. all the boundary operators are admissible morphisms of Banach A-modules.

Proof. This is [Ber90, Prop. 2.2.5]. □

Tate's acyclicity theorem implies in particular that $\widetilde{M}$ is a sheaf in the weak G-topology of $X = \mathcal{M}(A)$. This sheaf extends in a unique way to a sheaf of $\mathcal{O}_X$-modules in the analytic G-topology of X which we denote by $\widetilde{M}$ as well and which we call the sheaf *associated* to the A-module M.

Proposition 2.1.2. *Let $X = \mathcal{M}(A)$ be a k-affinoid space.*

(i) *The associated sheaf construction gives rise to a functor*

$$\mathrm{Mod}(A) \to \mathrm{Mod}(\mathcal{O}_X), \qquad M \mapsto \widetilde{M}.$$

(ii) *The functor $M \mapsto \widetilde{M}$ is left adjoint to the global sections functor $\mathcal{F} \mapsto \mathcal{F}(X)$. The unit of this adjunction is given by*

$$\eta_M : M \to \widetilde{M}(X) = A \otimes_A M, \qquad m \mapsto 1 \otimes m,$$

the counit is given on affinoid domains U by

$$\varepsilon_{\mathcal{F}} : \mathcal{O}_X(U) \otimes_A \mathcal{F}(X) \to \mathcal{F}(U), \qquad f \otimes m \mapsto f \cdot m|_U.$$

The unit $\eta_M : M \to \widetilde{M}(X)$ is an isomorphism.

(iii) *The functor $M \mapsto \widetilde{M}$ is fully faithful.*

(iv) *The functor $M \mapsto \widetilde{M}$ is cocontinuous, i.e. it commutes with arbitrary colimits.*

(v) *The functor $M \mapsto \widetilde{M}$ is a strong monoidal functor, i.e. it commutes with tensor products and we have $\widetilde{A} \cong \mathcal{O}_X$.*

(vi) *The functor $M \mapsto \widetilde{M}$ is faithfully exact, i.e. a sequence $M \to N \to P$ of A-modules is exact if and only if the associated sequence of $\mathcal{O}_X$-modules $\widetilde{M} \to \widetilde{N} \to \tilde{P}$ is exact.*

Proof. The statements all hold for completely formal reasons, except for maybe the left exactness of the functor $M \mapsto \widetilde{M}$. This follows from the flatness of the restriction morphisms $\mathcal{O}_X(X) \to \mathcal{O}_X(U)$ for affinoid domains $U \subseteq X$; see [Ber90, Prop. 2.2.4 (ii)]. □

Observe that a sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$ is associated to some A-module M (i.e. is isomorphic to a sheaf of the form $\widetilde{M}$) if and only if the counit $\varepsilon_{\mathcal{F}} : \widetilde{\mathcal{F}(X)} \to \mathcal{F}$ is an isomorphism. In this case we necessarily have $M \cong \mathcal{F}(X)$. The property of being associated to some finitely generated module can also be expressed purely in the monoidal category of sheaves of $\mathcal{O}_X$-modules, namely as the existence of a global finite presentation:

Proposition 2.1.3. *Let $X = \mathcal{M}(A)$ be a k-affinoid space and let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules. The sheaf $\mathcal{F}$ is associated to a finite A-module if and only if there exists an exact sequence of the form*

$$\mathcal{O}_X^m \to \mathcal{O}_X^n \to \mathcal{F} \to 0.$$

Proof. Let M be a finite A-module. Because A is Noetherian, there exists a finite presentation

$$A^m \to A^n \to M \to 0.$$

The associated sequence of $\mathcal{O}_X$-modules now has the form $\mathcal{O}_X^m \to \mathcal{O}_X^n \to \mathcal{F} \to 0$.

Conversely, let $\mathcal{O}_X^m \to \mathcal{O}_X^n \to \mathcal{F} \to 0$ be a global finite presentation of $\mathcal{F}$. The morphism $\mathcal{O}_X^m \to \mathcal{O}_X^n$ is associated to some morphism $A^m \to A^n$. Setting $M := \operatorname{Coker}(A^m \to A^n)$, we have $\mathcal{F} \cong \widetilde{M}$. □

2.2 Coherent sheaves

In this section, we assume for simplicity that all occurring G-topological spaces X have the property that X is an admissible open subset of itself. All concepts introduced below do however also work for general ringed G-topological spaces or, still more generally, for ringed topoi.

Definition 2.2.1. Let X be a ringed G-topological space and let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules.

(i) The sheaf $\mathcal{F}$ is called *finitely generated* if there exists an admissible covering $\mathfrak{U}$ of X such that, for every $U \in \mathfrak{U}$, there exists an exact sequence of the form

$$\mathcal{O}_X^n|_U \to \mathcal{F}|_U \to 0.$$

(ii) The sheaf $\mathcal{F}$ is called *finitely presented* if there exists an admissible open covering $\mathfrak{U}$ of X such that, for every $U \in \mathfrak{U}$, there exists an exact sequence of the form

$$\mathcal{O}_X^m|_U \to \mathcal{O}_X^n|_U \to \mathcal{F} \to 0.$$

(iii) The sheaf $\mathcal{F}$ is called *coherent* if it is finitely generated and if for every admissible open subset $U \subseteq X$, as well as for every morphism

$$\psi\colon \mathcal{O}_X^n|_U \to \mathcal{F}|_U,$$

the kernel $\mathrm{Ker}(\psi)$ is also finitely generated.

We collect here some of the basic formal properties of the above notions. Finite generation, finite presentation and coherence are local properties in the following sense:

Lemma 2.2.2. *Let X be a ringed G-topological space and let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules.*

(i) *If $\mathcal{F}$ is finitely generated, resp. finitely presented, resp. coherent, then for every admissible open subset $U \subseteq X$, the restriction $\mathcal{F}|_U$ is also finitely generated, resp. finitely presented, resp. coherent.*

(ii) *If $\mathfrak{U}$ is an admissible covering of X and $\mathcal{F}|_U$ is finitely generated, resp. finitely presented, resp. coherent for every $U \in \mathfrak{U}$, then $\mathcal{F}$ is also finitely generated, resp. finitely presented, resp. coherent.*

Lemma 2.2.3. *Let X be a ringed G-topological space. Every coherent sheaf on X is finitely presented and every finitely presented sheaf on X is finitely generated.*

Lemma 2.2.4. *Let X be a ringed G-topological space. The class of coherent sheaves of $\mathcal{O}_X$-modules is closed under kernels, cokernels, short exact sequences (in particular finite direct sums) and tensor products.*

Proof. For the closedness under kernels, cokernels and short exact sequences, we refer to [Sta19, Lem. 01BY] where the case of ringed topological spaces is treated, which, however, easily carries over to our more general situation. Regarding tensor products we can argue as follows:

Let $\mathcal{F}$ and $\mathcal{G}$ be two coherent sheaves of $\mathcal{O}_X$-modules. Then $\mathcal{F}$ is finitely presented (and this, in fact, suffices). We may work locally and assume that there exists a short exact sequence $\mathcal{O}_X^m \to \mathcal{O}_X^n \to \mathcal{F} \to 0$. Tensoring with $\mathcal{G}$ yields a short exact sequence $\mathcal{G}^m \to \mathcal{G}^n \to \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G} \to 0$ and the coherence of $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$ follows, because coherence is closed under direct sums and cokernels. □

Lemma 2.2.5. *Let X be a set and let T, T' be two G-topologies on X, and assume that T' is slightly finer than T. Furthermore, let $\mathcal{O}_T$ be a sheaf of rings on (X, T) and let $\mathcal{O}_{T'}$ be its extension to (X, T'). Then the equivalence*

$$\mathrm{Mod}(\mathcal{O}_{T'}) \xrightarrow{\sim} \mathrm{Mod}(\mathcal{O}_T)$$

from Proposition 1.1.5 restricts to an equivalence of the respective full subcategories of finitely generated, resp. finitely presented, resp. coherent sheaves.

On affinoid spaces, sheaves associated to finite modules are coherent, and we will see later in Theorem 2.5.1 that in fact every coherent sheaf on an affinoid space is associated to some finite module.

Lemma 2.2.6. *Let $X = \mathcal{M}(A)$ be a k-affinoid space and let M be a finite A-module. Then the associated sheaf $\widetilde{M}$ of $\mathcal{O}_X$-modules is coherent.*

Proof. By Lemma 2.2.5, it suffices to view $\widetilde{M}$ as a sheaf in the weak G-topology of X. From Proposition 2.1.3 it follows that $\widetilde{M}$ is finitely presented and, in particular, finitely generated. Now let $U \subseteq X$ be an affinoid domain and let $\psi\colon \mathcal{O}_X^n|_U \to \widetilde{M}|_U$ be a morphism of sheaves. We have to show that $\mathrm{Ker}(\psi)$ is finitely generated. We may assume that $U = X$. The morphism ψ is then associated to a homomorphism $\alpha\colon A^n \to M$. As A is Noetherian, $\mathrm{Ker}(\alpha)$ is finite. But $\mathrm{Ker}(\psi)$ is associated to $\mathrm{Ker}(\alpha)$ and again by Proposition 2.1.3, $\mathrm{Ker}(\psi)$ is finitely generated. □

2.3 Coherent sheaves on analytic spaces

Let X be a k-analytic space and let $\mathfrak{U}$ be an admissible covering of X by affinoid subdomains. We call a sheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules $\mathfrak{U}$-coherent if for every $U \in \mathfrak{U}$, the sheaf $\mathcal{F}|_U$ is associated to a finite $\mathcal{O}_X(U)$-module.

Proposition 2.3.1. *Let X be a k-analytic space and $\mathcal{F}$ a sheaf of $\mathcal{O}_X$-modules. Then the following are equivalent:*

(i) *The sheaf $\mathcal{F}$ is coherent.*

(ii) *The sheaf $\mathcal{F}$ is finitely presented.*

(iii) *The sheaf $\mathcal{F}$ is $\mathfrak{U}$-coherent for some admissible covering $\mathfrak{U}$ of X by affinoid subdomains.*

Proof. The implication "(i) $\Rightarrow$ (ii)" follows from Lemma 2.2.3.

So let us assume that the sheaf $\mathcal{F}$ is finitely presented. Then there exists an admissible covering $\mathfrak{U}$ of X such that, for every $U \in \mathfrak{U}$, there exists a short exact sequence of the form $\mathcal{O}_X^m|_U \to \mathcal{O}_X^n|_U \to \mathcal{F}|_U \to 0$. After refining the covering if necessary, we may assume that the covering subsets $U \in \mathfrak{U}$ are affinoid domains. Then Proposition 2.1.3 shows that the sheaves $\mathcal{F}|_U$ are associated to finite $\mathcal{O}_X(U)$-modules. This proves "(ii) $\Rightarrow$ (iii)".

In order to prove the remaining implication "(iii) $\Rightarrow$ (i)", we observe that by Lemma 2.2.6, the sheaves $\mathcal{F}|_U$ are coherent. The coherence of $\mathcal{F}$ therefore follows from Lemma 2.2.2. □

From Theorem 2.5.1 we will later be able to conclude that a coherent sheaf $\mathcal{F}$ is in fact $\mathfrak{U}$-coherent for every admissible covering $\mathfrak{U}$ of X by affinoid subdomains.

Lemma 2.3.2. *Let X be a k-analytic space and let $\mathfrak{U}$ be an admissible affinoid covering of X. Then the full subcategory of $\mathfrak{U}$-coherent sheaves of $\mathcal{O}_X$-modules is closed under kernels, cokernels, finite direct sums and tensor products.*

Proof. This follows immediately from Proposition 2.1.2. □

Lemma 2.3.3. *Let X be a k-analytic space and let $\mathfrak{U}$, $\mathfrak{V}$ be two admissible coverings of X by affinoid domains and let $\mathfrak{V}$ be a refinement of $\mathfrak{U}$. Then every $\mathfrak{U}$-coherent sheaf is also $\mathfrak{V}$-coherent.*

Corollary 2.3.4. *Let X be a k-analytic space. Given finitely many coherent sheaves $\mathcal{F}_1, \dots, \mathcal{F}_n$ of $\mathcal{O}_X$-modules, there exists an admissible affinoid covering $\mathfrak{U}$ of X such that each of the sheaves $\mathcal{F}_1, \dots, \mathcal{F}_n$ is $\mathfrak{U}$-coherent.*

Proof. For each $i \in [1 \,..\, n]$, there exists an admissible affinoid covering $\mathfrak{U}_i$ such that $\mathcal{F}_i$ is $\mathfrak{U}_i$-coherent. Then $\mathfrak{U}_1 \cap \cdots \cap \mathfrak{U}_n \coloneqq \{ U_1 \cap \cdots \cap U_n \mid U_i \in \mathfrak{U}_i \}$ is an admissible covering of X refining each $\mathfrak{U}_i$. Writing each $U \in \mathfrak{U}_1 \cap \cdots \cap \mathfrak{U}_n$ as an admissible union of affinoid domains, we obtain an admissible affinoid covering $\mathfrak{U}$ of X refining each of the coverings $\mathfrak{U}_i$. Then each $\mathcal{F}_i$ is $\mathfrak{U}$-coherent. □

2.4 Scalar extension of coherent sheaves

Let X be a k-analytic space and let $k \hookrightarrow l$ be an extension of complete non-archimedean fields. As explained in Section 1.5, there is a morphism of ringed G-topological spaces $\pi\colon\ l \hat{\otimes}_k X \to X$, restricting for affinoid domains $U = \mathcal{M}(A) \subseteq X$ to a morphism $l \hat{\otimes}_k U = \mathcal{M}(l \hat{\otimes}_k A) \to U = \mathcal{M}(A)$ induced by the morphism $A \to l \hat{\otimes}_k A$ of Banach rings.

If $\mathcal{F}$ is a coherent sheaf of modules on X, then we will use the notation $l \hat{\otimes}_k \mathcal{F}$ for the inverse image sheaf $\pi^*(\mathcal{F})$ on $l \hat{\otimes}_k X$. This notation is justified by the following lemma.

Lemma 2.4.1. *Let $X = \mathcal{M}(A)$ be a k-affinoid space, $k \hookrightarrow l$ an extension of complete non-archimedean fields and M a finite A-module. We equip M with the unique structure of a finite Banach A-module. Then we have a canonical isomorphism of sheaves of $\mathcal{O}_{l \hat{\otimes}_k X}$-modules*

$$l \hat{\otimes}_k \widetilde{M} \cong \widetilde{l \hat{\otimes}_k M}.$$

Proof. We choose an admissible finite presentation $A^m \to A^n \to M \to 0$ of M as a Banach A-module. This expresses that M is a cokernel of $A^m \to A^n$ in the category of Banach A-modules. Since the completed tensor product is left adjoint (to the Hom-Banach module), the sequence

$$(l \hat{\otimes}_k A)^m \to (l \hat{\otimes}_k A)^n \to l \hat{\otimes}_k M \to 0$$

is admissible and right exact. Applying the associated sheaf functor, we obtain an exact sequence

$$\mathcal{O}^m_{l \hat{\otimes}_k X} \to \mathcal{O}^n_{l \hat{\otimes}_k X} \to \widetilde{l \hat{\otimes}_k M} \to 0.$$

On the other hand, applying to $A^m \to A^n \to M \to 0$ the associated sheaf functor, we obtain an exact sequence

$$\mathcal{O}_X^m \to \mathcal{O}_X^n \to \widetilde{M} \to 0.$$

Since the functor $\pi^* = (\mathcal{F} \mapsto l \mathbin{\hat{\otimes}_k} \mathcal{F})$, as a monoidally left adjoint functor, commutes with the monoidal unit, coproducts and cokernels, we obtain the exact sequence

$$\mathcal{O}^m_{l\hat{\otimes}_k X} \to \mathcal{O}^n_{l\hat{\otimes}_k X} \to l \mathbin{\hat{\otimes}_k} \widetilde{M} \to 0.$$

Hence we see that $l \mathbin{\hat{\otimes}_k} \widetilde{M}$ and $\widetilde{l \mathbin{\hat{\otimes}_k} M}$ are cokernels of the same morphism, whence isomorphic. □

Remark 2.4.2. Consider a cartesian square

$$\begin{tikzcd} X' \arrow[r] \arrow[d, "\varphi'"'] & Y' = \mathcal{M}(B') \arrow[d, "\varphi"] \\ X \arrow[r] & Y = \mathcal{M}(B) \end{tikzcd}$$

of k-analytic spaces and let $\mathcal{F}$ be a sheaf of coherent $\mathcal{O}_X$-modules on X. Then we denote the pullback $\varphi'^*(\mathcal{F})$ by $B' \mathbin{\hat{\otimes}_B} \mathcal{F}$. This is legitimate, because if $X = \mathcal{M}(A)$ is affinoid and $\mathcal{F} = \widetilde{M}$ for some finite A-module M, then as above one can show that $B' \mathbin{\hat{\otimes}_B} \widetilde{M} \cong \widetilde{B' \mathbin{\hat{\otimes}_B} M}$.

Corollary 2.4.3. *Let X be a k-analytic space and let $k \hookrightarrow l$ be an extension of complete non-archimedean fields. If $\mathcal{F}$ is a sheaf of coherent $\mathcal{O}_X$-modules, then the scalar extension $l \mathbin{\hat{\otimes}_k} \mathcal{F}$ is again coherent.*

Proof. Using Proposition 2.3.1, we may assume that $X = \mathcal{M}(A)$ is an affinoid space and that $\mathcal{F}$ is associated to a finite A-module. Now the claim follows from Lemma 2.4.1.

Alternatively, one can show for general morphisms of ringed G-topological spaces that inverse images of finitely presented sheaves are finitely presented, and then again use Proposition 2.3.1. □

Corollary 2.4.4. *Let X be a k-analytic space and let $r = (r_1, \dots, r_s)$ be a family of positive real numbers which are $\mathbb{Q}$-linearly independent in $\mathbb{R}_{>0}/\sqrt{|k^\times|}$. Then the scalar extension functor for coherent sheaves*

$$\mathrm{Coh}(\mathcal{O}_X) \to \mathrm{Coh}(\mathcal{O}_{k_r \hat{\otimes}_k X}), \qquad \mathcal{F} \mapsto k_r \hat{\otimes}_k \mathcal{F}$$

is faithfully exact, i.e. a sequence $\mathcal{F} \to \mathcal{G} \to \mathcal{H}$ of coherent sheaves of $\mathcal{O}_X$-modules is exact if and only if the corresponding sequence $k_r \hat{\otimes}_k \mathcal{F} \to k_r \hat{\otimes}_k \mathcal{G} \to k_r \hat{\otimes}_k \mathcal{H}$ is exact.

Proof. Locally, we may assume by Corollary 2.3.4 that $X = \mathcal{M}(A)$ is affinoid and that all sheaves $\mathcal{F}$, $\mathcal{G}$ and $\mathcal{H}$ are associated to finite A-modules. Now the claim follows from [Ber90, Prop. 2.1.2 (ii) and Prop. 2.1.10 (iii)]. □

2.5 Kiehl's theorem

This section is devoted to a proof of the following theorem:

Theorem 2.5.1 (Kiehl)**.** *Let $X = \mathcal{M}(A)$ be a k-affinoid space and let $\mathcal{F}$ be a coherent sheaf of $\mathcal{O}_X$-modules. Then $\mathcal{F}$ is associated to a finite A-module.*

Let us first of all note the following slightly stronger formulation which, however, follows immediately from Theorem 2.5.1 and Proposition 2.1.2.

Corollary 2.5.2. *Let $X = \mathcal{M}(A)$ be a k-affinoid space. Then the two functors*

$$\mathrm{FinMod}(A) \to \mathrm{Coh}(\mathcal{O}_X), \qquad M \mapsto \widetilde{M}$$

and

$$\mathrm{Coh}(\mathcal{O}_X) \to \mathrm{FinMod}(A), \qquad \mathcal{F} \mapsto \mathcal{F}(X)$$

are mutually inverse equivalences of categories.

Note that a priori it is not even clear that the latter functor is well-defined, i.e. that the module of global sections of a coherent sheaf $\mathcal{F}$ is finitely generated.

Let $\mathcal{F}$ be a coherent sheaf of $\mathcal{O}_X$-modules on the k-affinoid space $X = \mathcal{M}(A)$. By Proposition 2.3.1, there exists an admissible affinoid covering $\mathfrak{U}$ of $\mathcal{F}$ such that $\mathcal{F}$ is $\mathfrak{U}$-coherent. Since k-affinoid spaces are compact, we may assume by Lemma 1.2.3 that this covering is finite.

For each affinoid domain $U \subseteq X$ which is contained in one of the U_i, the module $\mathcal{F}(U)$ is by assumption finite over $\mathcal{O}_X(U_i)$ and, hence, carries a unique structure as a finite Banach $\mathcal{O}_X(U_i)$-module. Let us now consider the augmented Čech complex of $\mathcal{F}$ with respect to the covering $\mathfrak{U}$:

$$0 \to \mathcal{F}(X) \to \prod_{i \in I} \mathcal{F}(U_i) \to \prod_{i,j \in I} \mathcal{F}(U_i \cap U_j) \to \dots .$$

Beginning with $\prod_{i \in I} \mathcal{F}(U_i)$, all the modules carry the structure of Banach k-vector spaces if we equip $\mathcal{F}(U)$ with norms as above, and the finite products with the maximum norm. Since $\mathcal{F}(X) \hookrightarrow \prod_{i \in I} \mathcal{F}(U_i)$ is injective with closed image by the sheaf condition for $\mathcal{F}$, we can equip $\mathcal{F}(X)$ with the subspace norm inherited from $\prod_{i \in I} \mathcal{F}(U_i)$. In this way, the augmented Čech complex is a complex of Banach k-vector spaces.

If we knew already that Kiehl's theorem holds, i.e. that $\mathcal{F}$ is globally associated to a finite A-module, then Tate's Theorem 2.1.1 would imply that the augmented Čech complex is admissible and exact. We now want to establish this result directly, using only the $\mathfrak{U}$-coherence of $\mathcal{F}$ and afterwards use it for a proof of Kiehl's theorem.

Proposition 2.5.3. *Let $X = \mathcal{M}(A)$ be a k-affinoid space, $\mathfrak{U}$ a finite admissible exact covering of X and $\mathcal{F}$ a $\mathfrak{U}$-coherent sheaf of $\mathcal{O}_X$-modules. Then the augmented Čech complex of $\mathcal{F}$ with respect to $\mathfrak{U}$ is admissible and exact (with norms as explained above).*

Proof. First, we are going to reduce to the strictly k-affinoid situation. For this, we choose a field extension $k \hookrightarrow k_r$ in such a way that k_r is non-trivially valued and such that both the space $k_r \hat{\otimes}_k X$ as well as all of the affinoid domains $k_r \hat{\otimes}_k U$ with $U \in \mathfrak{U}$ are strictly k_r-affinoid. By [Ber90, Prop. 2.1.2 (ii)], it suffices to prove that the completed tensor product complex

$$0 \to k_r \hat{\otimes}_k \mathcal{F}(X) \to \prod_{i \in I} k_r \hat{\otimes}_k \mathcal{F}(U_i) \to \prod_{i,j \in I} k_r \hat{\otimes}_k \mathcal{F}(U_i \cap U_j) \to \dots$$

is admissible and exact. But this complex is precisely the Čech complex of the $k_r \hat{\otimes}_k \mathfrak{U}$-coherent sheaf $k_r \hat{\otimes}_k \mathcal{F}$ of $\mathcal{O}_{k_r \hat{\otimes}_k X}$-modules. Therefore, we may assume that k is non-trivially valued and that the space X and all the affinoid subdomains $U \in \mathfrak{U}$ are strictly affinoid. In this case, admissibility follows from the exactness by Banach's open mapping theorem, so all that is left to show is exactness of the complex.

An inspection of the proof of [BGR84, Prop. 8.2.2/5] shows that we may assume that $\mathfrak{U} = \{X\{f\}, X\{f^{-1}\}\}$ (we cannot apply this proposition directly, because we have to make sure that in every reduction step the assumption of $\mathfrak{U}$-coherence survives). We compute the Čech cohomology groups using alternating cochains. Then we see in particular, that the complex is acyclic in degrees $q \geq 2$.

We introduce the following notation: $U_1 := X\{f\}$, $U_2 := X\{f^{-1}\}$, $U_{12} := U_1 \cap U_2 = X\{f, f^{-1}\}$, furthermore $A_1 := \mathcal{O}_X(U_1)$, $A_2 := \mathcal{O}_X(U_2)$ and $A_{12} := \mathcal{O}_X(U_{12})$, as well as $M_1 := \mathcal{F}(U_1)$, $M_2 := \mathcal{F}(U_2)$ and $M_{12} := \mathcal{F}(U_{12})$. It is enough to prove the surjectivity of the map

$$d^0 : M_1 \oplus M_2 \to M_{12}, \qquad (u^+, u^-) \mapsto u^+|_{U_{12}} - u^-|_{U_{12}}.$$

By assumption, we can find finite generating systems $v'_1, \dots, v'_m$ of M_1 as an A_1-module as well as $w'_1, \dots, w'_n$ of M_2 as an A_2-module. We write $v_1, \dots, v_m \in M_{12}$ and $w_1, \dots, w_n \in M_{12}$ for the respective restrictions to U_{12}. As $\mathcal{F}$ is $\mathfrak{U}$-coherent, both $v_1, \dots, v_m$ and $w_1, \dots, w_n$ are generating systems of M_{12} as an A_{12}-module. We consider M_1, M_2 and M_{12} as Banach modules with respect to the residue norms inherited from the maps

$$A_1^m \twoheadrightarrow M_1, \qquad A_2^n \twoheadrightarrow M_2, \qquad A_{12}^m \twoheadrightarrow M_{12}$$

induced by $v'_1, \dots, v'_m$ and $w'_1, \dots, w'_n$ and $v_1, \dots, v_m$, respectively. By a standard approximation argument, it suffices to prove the following claim:

Claim. *Given $\varepsilon > 0$, there exists a constant $\alpha > 1$ such that for every $u \in M_{12}$, there exist two elements $u^+ \in M_1$ and $u^- \in M_2$ such that*

$$|u^+| \leq \alpha|u|, \qquad |u^-| \leq \alpha|u|, \qquad |u - u^+|_{U_{12}} - u^-|_{U_{12}}| \leq \varepsilon|u|.$$

So suppose that $\varepsilon > 0$ is given. We also fix some constant $\beta > 1$. Since both the v_i and the w_j generate the module M_{12}, we can find

$c_{ij}, d_{jl} \in A_{12}$ such that

$$\begin{aligned} v_i &= \sum_{j=1}^{n} c_{ij} w_j, && i \in [1 \,..\, n] \\ w_j &= \sum_{l=1}^{m} d_{jl} v_l, && j \in [1 \,..\, n]. \end{aligned}$$

Since $U_{12} \hookrightarrow U_2$ is a Weierstraß domain, the map $A_2 \to A_{12}$ has dense image; whence there exist $c'_{ij} \in A_2$ such that

$$\max_{ijl} |c_{ij} - c'_{ij}|_{U_{12}}| |d_{jl}| \leq \beta^{-2}\varepsilon.$$

I claim that now $\alpha := \beta^2 \max_{ij}(|c'_{ij}| + 1)$ satisfies our requirements.

Indeed, suppose that $u \in M_{12}$ is given. As the norm on M_{12} is induced from the map $A_{12}^m \twoheadrightarrow M_{12}$, we can find a tuple $a_1, \dots, a_m \in A_{12}^m$ such that $u = \sum_{i=1}^m a_i v_i$ and such that $\max_i a_i \leq \beta |u|$. By a similar argument (applied to $A_1 \oplus A_2 \twoheadrightarrow A_{12}$), we can write

$$a_i = a_i^+|_{U_{12}} + a_i^-|_{U_{12}}$$

with elements $a_i^+ \in A_1$, $a_i^- \in A_2$ satisfying $|a_i^+| \leq \beta |a_i|$ and $|a_i^-| \leq \beta |a_i|$. Now we define

$$\begin{aligned} u^+ &:= \sum_{i=1}^{m} a_i^+ v'_i \in M_1, \\ u^- &:= \sum_{i=1}^{m} a_i^- \Big(\sum_{j=1}^{n} c'_{ij} w'_j\Big) \in M_2. \end{aligned}$$

Then we have

$$\begin{aligned} |u^+| &\leq \max_i |a_i^+| \leq \max_i \beta |a_i| \leq \beta^2 |u| \leq \alpha |u|, \\ |u^-| &\leq \max_{ij} |a_i^-| |c'_{ij}| \leq \max_i \beta |a_i| \max_{ij} |c'_{ij}| \leq \beta^2 |u| \max_{ij} |c'_{ij}| \leq \alpha |u|, \end{aligned}$$

as well as

$$\begin{aligned} u &= \sum_{i=1}^{m} (a_i^+ + a_i^-) v_i \\ &= u^+ + \sum_{i=1}^{m} \sum_{j=1}^{n} a_i^- c_{ij} w_j \\ &= u^+ + u^- + \sum_{i=1}^{m} \sum_{j=1}^{n} a_i^- (c_{ij} - c'_{ij}) w_j. \end{aligned}$$

(Here we suppress the respective restrictions to U_{12} in the notation.) Therefore we have

$$\begin{aligned} |u - u^+ - u^-| &= |\sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{l=1}^{m} a_i^- (c_{ij} - c'_{ij}) d_{jl} v_l| \\ &\leq \max_{ijl} |a_i^-||c_{ij} - c'_{ij}||d_{jl}| \\ &\leq \beta^2 |u| \beta^{-2} \varepsilon \\ &= \varepsilon |u|. \end{aligned}$$

This proves the claim and, hence, the proposition. □

This form of cohomological triviality suffices to imply exactness of the global section functor in the following sense:

Corollary 2.5.4. *Let $X = \mathcal{M}(A)$ be a k-affinoid space, $\mathfrak{U}$ a finite affinoid covering of X and $\psi : \mathcal{F} \to \mathcal{F}'$ an epimorphism of $\mathfrak{U}$-coherent sheaves of $\mathcal{O}_X$-modules. Then the map $\psi(X) : \mathcal{F}(X) \to \mathcal{F}'(X)$ on global sections is also an epimorphism.*

Proof. We set $\mathcal{F}'' := \operatorname{Ker}(\psi)$. Then the sequence $0 \to \mathcal{F}'' \to \mathcal{F} \to \mathcal{F}' \to 0$ is a short exact sequence of $\mathfrak{U}$-coherent sheaves of $\mathcal{O}_X$-modules. For every U which is contained in one of the U_i, the sequence $0 \to \mathcal{F}''(U) \to \mathcal{F}(U) \to \mathcal{F}'(U) \to 0$ is exact by Proposition 2.1.2. In particular, this is true if U is a higher intersection of sets of $\mathfrak{U}$. Therefore, the sequence of cochain complexes

$$0 \to C^\bullet(\mathfrak{U}, \mathcal{F}'') \to C^\bullet(\mathfrak{U}, \mathcal{F}) \to C^\bullet(\mathfrak{U}, \mathcal{F}') \to 0$$

is exact. Since $H^1(\mathfrak{U}, \mathcal{F}'') = 0$, the long exact cohomology sequence implies the exactness of $\mathcal{F}(X) \to \mathcal{F}'(X) \to 0$. □

We prove the following lemma only in the strictly affinoid case, but of course it is true also in the general affinoid case, as will follow once we finish our proof of Kiehl's theorem.

Lemma 2.5.5. *Let $X = \mathcal{M}(A)$ be a strictly k-affinoid space, $\mathfrak{U}$ a finite strictly affinoid covering of X and $\mathcal{F}$ a $\mathfrak{U}$-coherent sheaf of $\mathcal{O}_X$-modules. Then for some $n \in \mathbb{N}$, there exists an epimorphism $\mathcal{O}_X^n \twoheadrightarrow \mathcal{F}$.*

Proof. We write $A_j := \mathcal{O}_X(U_j)$ as well as $M_j := \mathcal{F}(U_j)$, so that $\mathcal{F}|_{U_j}$ is by assumption associated to the finite A_j-module M_j. It suffices to prove the following claim:

Claim 1. *For each $j \in J$, the map $A_j \otimes_A \mathcal{F}(X) \to M_j$ is surjective.*

Indeed, this implies the lemma as follows: The claim expresses that M_j as an A_j-module is generated by the image of the restriction map $\mathcal{F}(X) \to M_j$. Since M_j is Noetherian, there exist, in fact, finitely many elements of $\mathcal{F}(X)$ whose restrictions to U_j generate the module M_j. Since there are only finitely many U_j involved, we obtain finitely many global sections $s_1, \dots, s_n \in \mathcal{F}(X)$ whose respective restrictions generate each of the modules M_j. The morphism $\mathcal{O}_X^n \to \mathcal{F}$ which is defined by these elements is an epimorphism, as this can be checked locally on the U_j.

If $x \in X$ is a Zariski-closed point and $\mathfrak{m}_x \in \operatorname{MSpec}(A)$ is the corresponding maximal ideal, then we can consider the quotient sheaf $\mathcal{F} \twoheadrightarrow \mathcal{F}/\mathfrak{m}_x\mathcal{F} := \operatorname{Coker}(\widetilde{\mathfrak{m}_x} \otimes_{\mathcal{O}_X} \mathcal{F} \to \mathcal{F})$. This sheaf is $\mathfrak{U}$-coherent by Lemma 2.3.2. In fact, it is associated to the module $M_j/\mathfrak{m}_x M_j$ on U_j.

Claim 2. *For every $x \in \operatorname{MSpec}(A) \cap U_j \subseteq X$, the restriction morphism*

$$(\mathcal{F}/\mathfrak{m}_x\mathcal{F})(X) \to (\mathcal{F}/\mathfrak{m}_x\mathcal{F})(U_j)$$

is bijective.

This claim indeed implies Claim 1. To see this, we consider the following commutative diagram of A_j-modules:

$$\begin{tikzcd} A_j \otimes_A \mathcal{F}(X) \arrow[r] \arrow[d] & A_j \otimes_A (\mathcal{F}/\mathfrak{m}_x\mathcal{F})(X) \arrow[d] \\ M_j \arrow[r] & (\mathcal{F}/\mathfrak{m}_x\mathcal{F})(U_j) = M_j/\mathfrak{m}_x M_j = M_j/(A_j\mathfrak{m}_x)M_j. \end{tikzcd}$$

Corollary 2.5.4 and the right exactness of the tensor product imply the surjectivity of the upper map. Claim 2, which we still have to prove, implies the surjectivity of the right map. It follows that the composition is surjective. Since $A_j\mathfrak{m}_x$ runs through all maximal ideals of A_j, the surjectivity of $A_j \otimes_A \mathcal{F}(X) \to M_j$ follows from Nakayama's lemma.

It remains to prove Claim 2. In order to do this, we consider the following diagram with exact rows:

$$\begin{tikzcd} (\mathcal{F}/\mathfrak{m}_x\mathcal{F})(X) \arrow[r] \arrow[d] & \prod_{i\in I}(\mathcal{F}/\mathfrak{m}_x\mathcal{F})(U_i) \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & \prod_{i,i'\in I}(\mathcal{F}/\mathfrak{m}_x\mathcal{F})(U_{ii'}) \arrow[d] \\ (\mathcal{F}/\mathfrak{m}_x\mathcal{F})(U_j) \arrow[r] & \prod_{i\in I}(\mathcal{F}/\mathfrak{m}_x\mathcal{F})(U_{ij}) \arrow[r, shift left] \arrow[r, shift right] & \prod_{i,i'\in I}(\mathcal{F}/\mathfrak{m}_x\mathcal{F})(U_{ii'j}). \end{tikzcd}$$

It follows that it suffices to prove that $(\mathcal{F}/\mathfrak{m}_x\mathcal{F})(U') \to (\mathcal{F}/\mathfrak{m}_x\mathcal{F})(U' \cap U_j)$ is bijective whenever $x \in \operatorname{MSpec}(A) \cap U_j$ and $U' \subseteq X$ is an affinoid domain contained in some $U \in \mathfrak{U}$. Then $\mathcal{F}|_{U'}$ is restricted to a finite A'-module M'. We write $A'_j := \mathcal{O}_X(U' \cap U_j)$. Then $(\mathcal{F}/\mathfrak{m}_x\mathcal{F})|_{U'}$ is associated to the A'-module $M'/\mathfrak{m}_x M'$. The canonical map

$$M'/\mathfrak{m}_x M' \to A'_j \otimes_{A'} M'/\mathfrak{m}_x M' \cong A'_j/\mathfrak{m}_x A'_j \otimes_{A'/\mathfrak{m}_x A'} M'/\mathfrak{m}_x M'$$

is bijective for all $x \in \operatorname{MSpec}(A) \cap U_j$, since $A'/\mathfrak{m}_x A' \to A'_j/\mathfrak{m}_x A'_j$ is an isomorphism (if $x \in U' \cap U_j$, then both rings coincide with $\mathcal{H}(x)$, and otherwise, both rings are trivial). □

Proposition 2.5.6. *Let $X = \mathcal{M}(A)$ be a k-affinoid space, $\mathfrak{U}$ a finite affinoid covering of X and $\mathcal{F}$ a $\mathfrak{U}$-coherent sheaf of $\mathcal{O}_X$-modules. Then $\mathcal{F}$ is associated to a finite A-module.*

Proof. If X and $\mathfrak{U}$ are strictly affinoid, then this follows easily from Lemma 2.5.5: By this lemma, there exists an epimorphism $\mathcal{O}_X^n \twoheadrightarrow \mathcal{F}$. Its kernel is again $\mathfrak{U}$-coherent and by another application of this lemma, we obtain an exact sequence $\mathcal{O}_X^m \to \mathcal{O}_X^n \to \mathcal{F} \to 0$. Now the claim follows from Proposition 2.1.3.

It remains to reduce the general case to the strict situation. If X and $\mathfrak{U}$ are not necessarily strictly k-affinoid, then we choose $k \hookrightarrow k_r$ such that $k_r \hat{\otimes}_k X$ and $k_r \hat{\otimes}_k \mathfrak{U}$ are strictly k_r-affinoid. We write $A_i := \mathcal{O}_X(U_i)$, $A_{ij} := \mathcal{O}_X(U_i \cap U_j)$ as well as $M := \mathcal{F}(X)$, $M_i := \mathcal{F}(U_i)$ and $M_{ij} := \mathcal{F}(U_i \cap U_j)$. We also equip all the modules M_i, M_{ij} with structures of finite Banach modules and we equip M with the subspace norm with respect to the embedding $M \hookrightarrow \prod_{i\in I} M_i$.

From the strict case we know that the $k_r \hat{\otimes}_k \mathfrak{U}$-coherent sheaf $k_r \hat{\otimes}_k \mathcal{F}$ of $\mathcal{O}_{k_r \hat{\otimes}_k X}$-modules is associated to some finite $k_r \hat{\otimes}_k A$-module (namely its global sections $M_r := (k_r \hat{\otimes}_k \mathcal{F})(k_r \hat{\otimes}_k X)$). We equip the finite module M_r with its unique Banach module structure.

Claim. *The $k_r \hat{\otimes}_k A$-module M_r is canonically isomorphic to $k_r \hat{\otimes}_k M$ as a Banach module.*

To prove this claim, we note that by the sheaf condition for $\mathcal{F}$, we have an exact sequence

$$0 \to M \to \prod_{i\in I} M_i \to \prod_{i,j\in I} M_{ij}$$

which is admissible by Proposition 2.5.3. By [Ber90, Prop. 2.1.2 (ii)], the completed tensor product

$$0 \to k_r \hat{\otimes}_k M \to \prod_{i\in I} k_r \hat{\otimes}_k M_i \to \prod_{i,j\in I} k_r \hat{\otimes}_k M_{ij}$$

is also admissible and exact. However, the sheaf condition for $k_r \hat{\otimes}_k \mathcal{F}$ yields an admissible short exact sequence

$$0 \to M_r \to \prod_{i\in I} k_r \hat{\otimes}_k M_i \to \prod_{i,j\in I} k_r \hat{\otimes}_k M_{ij}.$$

It follows that both M_r and $k_r \hat{\otimes}_k M$ are kernels of the map $\prod_{i\in I} k_r \hat{\otimes}_k$

$M_i \to \prod_{i,j\in I} k_r \hat{\otimes}_k M_{ij}$ in the category of Banach $k_r \hat{\otimes}_k A$-modules, and, hence, that they are isomorphic.

From [Ber90, Prop. 2.1.11] it now follows that M is a finite Banach A-module. It remains to prove that the maps $A_i \otimes_A M = A_i \hat{\otimes}_A M \to M_i$ are isomorphisms of Banach modules. After completed scalar extension with k_r, these maps can be identified with $(k_r \hat{\otimes}_k A_i) \hat{\otimes}_{k_r \hat{\otimes}_k A} M_r \to k_r \hat{\otimes}_k M_i$. Since $k_r \hat{\otimes}_k \mathcal{F}$ is associated to the module M_r, these maps are isomorphisms. Another application of [Ber90, Prop. 2.1.2 (ii)] finishes the proof. □

Proposition 2.5.6 together with Proposition 2.3.1 now gives a complete proof of Kiehl's theorem.

2.6 Cohomology of coherent sheaves

Kiehl's theorem together with Tate's acyclicity theorem imply that coherent sheaves on affinoid spaces are cohomologically trivial. This allows to compute coherent sheaf cohomology on separated spaces using Čech methods, as in algebraic geometry.

Proposition 2.6.1. *Let $X = \mathcal{M}(A)$ be a k-affinoid space and let $\mathcal{F}$ be a sheaf of coherent $\mathcal{O}_X$-modules. Then $\mathcal{F}$ has no higher sheaf cohomology, i.e. we have*

$$H^q(X, \mathcal{F}) = 0$$

for $q > 0$.

Proof. Sheaf cohomology is the right derived functor of the global sections functor $\mathrm{Mod}(\mathcal{O}_X) \to \mathrm{Mod}(A)$. By Proposition 1.1.5, we may view $\mathcal{F}$ as a sheaf in the weak G-topology of X. By Kiehl's theorem, $\mathcal{F}$ is associated to a finite A-module. From Tate's theorem, we know that the Čech cohomology of $\mathcal{F}$ with respect to every admissible covering in the weak G-topology vanishes. Now the claim follows from [Sta19, Lem. 03F9]. □

Proposition 2.6.2. *Let X be a separated k-analytic space, let $\mathfrak{U}$ be an admissible affinoid covering of X and let $\mathcal{F}$ be a coherent sheaf of $\mathcal{O}_X$-modules. Then the sheaf cohomology of $\mathcal{F}$ can be computed as the Čech cohomology of $\mathcal{F}$ with respect to $\mathfrak{U}$:*

$$H^q(X, \mathcal{F}) \cong H^q(\mathfrak{U}, \mathcal{F}).$$

Proof. Since X is separated, all the higher intersections of sets from $\mathfrak{U}_i$ are again affinoid. Since the sheaf cohomology of $\mathcal{F}$ vanishes on affinoid subdomains by Proposition 2.6.1, the result follows from [Sta19, Lem. 03F7]. □

Remark 2.6.3. If X is a compact, separated k-analytic space, then X has a finite affinoid covering $\mathfrak{U} = (U_i)_{i \in I}$, and all the higher intersections of sets from $\mathfrak{U}$ are again affinoid. If $\mathcal{F}$ is a coherent sheaf of $\mathcal{O}_X$-modules, then by Kiehl's theorem, the restriction of $\mathcal{F}$ to each U_i is associated to a finite module. Hence, the Čech complex

$$0 \to \mathcal{F}(X) \to \prod_{i \in I} \mathcal{F}(U_i) \to \prod_{i,j \in I} \mathcal{F}(U_i \cap U_j) \to \dots$$

is, in fact, a complex of Banach k-vector spaces. It seems to be unknown when the coboundary operators of this complex are admissible maps of Banach spaces. If X is affinoid, then the complex is exact and, hence, admissible by Banach's theorem (as we noted in Theorem 2.1.1). If X is proper over some k-affinoid space, then for a certain covering $\mathfrak{V}$ of X we will prove in Theorem 3.1.1 that the Čech-complex is admissible.

If the Čech complex is admissible, then its coboundary operators have closed image (and for non-trivially valued k, this condition is, in fact, equivalent to admissibilty) and hence, the cohomology groups $H^q(X, \mathcal{F})$ carry the structure of Banach k-vector spaces, which can be very useful to know. For example, for a field extension $k \hookrightarrow k_r$, one can conclude that

$$H^q(k_r \hat{\otimes}_k X, k_r \hat{\otimes}_k \mathcal{F}) \cong k_r \hat{\otimes}_k H^q(X, \mathcal{F}),$$

using the exactness of the completed scalar extension along k_r.

2.7 Coherent sheaves on compact spaces

Coherent sheaves on compact analytic spaces behave similar to coherent sheaves on Noetherian schemes. For example, we have:

Proposition 2.7.1. *Let X be a compact k-analytic space. Then the category* $\mathrm{Coh}(\mathcal{O}_X)$ *of coherent sheaves of $\mathcal{O}_X$-modules is a Noetherian abelian category, i.e. whenever $\mathcal{F}$ is a coherent sheaf and $\mathcal{F}_1 \subseteq \mathcal{F}_2 \subseteq \cdots \subseteq \mathcal{F}$ is an ascending sequence of coherent subsheaves of $\mathcal{F}$, then this sequence becomes stationary at some $\mathcal{F}_i$.*[1]

Proof. By Lemma 1.2.3, there exists a finite affinoid covering $\mathfrak{U}$ of X. The equalities $\mathcal{F}_i = \mathcal{F}_{i+1}$ can be tested on each member of $\mathfrak{U}$, so we may assume that $X = \mathcal{M}(A)$ is affinoid. But then by Corollary 2.5.2, the claim translates into the Noetherianity of the category of finite A-modules, which holds, because A is a Noetherian ring. □

The following has well-known equivalents in commutative algebra. We don't try to go for full generality here, but formulate the concepts in the way we will use them in the next chapter. Let X be a compact k-analytic space and let $b \in \mathcal{O}_X(X)$ be some globally defined function. Let $\mathcal{F}$ be a coherent sheaf of $\mathcal{O}_X$-modules. A *b-filtration* of $\mathcal{F}$ is a sequence $\mathcal{F} = \mathcal{F}_0 \supseteq \mathcal{F}_1 \supseteq \ldots$ of coherent subsheaves of $\mathcal{F}$ such that $b \cdot \mathcal{F}_i \subseteq \mathcal{F}_{i+1}$ for all $i \in \mathbb{N}$. The b-filtration is called *b-stable* if there is some index $i_0 \in \mathbb{N}$ such that $\mathcal{F}_i = b^{i-i_0}\mathcal{F}_{i_0}$ for all $i \geq i_0$.

Proposition 2.7.2 (Artin-Rees). *Let X be a compact k-analytic space, let $\mathcal{F}$ be a sheaf of coherent $\mathcal{O}_X$-modules and let $\mathcal{G}$ be a coherent subsheaf of $\mathcal{F}$. Then the b-filtration*

$$\mathcal{G} = \mathcal{F} \cap \mathcal{G} \supseteq b \cdot \mathcal{F} \cap \mathcal{G} \supseteq b^2 \cdot \mathcal{F} \cap \mathcal{G} \supseteq \ldots$$

of $\mathcal{G}$ is b-stable.

[1]Sometimes, one also requires that a Noetherian category should be essentially small. This, however, is also satisfied in our situation, because locally, a coherent sheaf is determined by a presentation matrix.

Proof. Again, by considering a finite affinoid covering, this can be reduced to the affinoid situation, where it follows from the analogous fact of Noetherian commutative algebra [AM69, Cor. 10.10]. □

3 The proper mapping theorem

This chapter is devoted to a proof of Grauert's proper mapping theorem, which asserts the coherence of the higher direct images $R^q\varphi_*(\mathcal{F})$ of a coherent sheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules under a proper morphism $\varphi : X \to Y$ of k-analytic spaces. In rigid analytic geometry, this result is also due to Kiehl [Kie67a]. Because the statement of this theorem is local on the base space Y, we can always assume that $Y = \mathcal{M}(B)$ is affinoid, so that X is a compact, separated k-analytic space which has two finite affinoid coverings $\mathfrak{U} = (U_i)_{i\in I}$ and $\mathfrak{V} = (V_i)_{i\in I}$ such that $V_i \Subset_Y U_i$ for all $i \in I$. We follow the presentation of [Bos14] quite closely, giving the necessary additions to also treat the not necessarily strictly analytic case. The general procedure is the same as in the original work by Kiehl.

In Section 3.1, we are going to prove that if $\mathcal{F}$ is a coherent sheaf of $\mathcal{O}_X$-modules, then $H^q(X,\mathcal{F})$ is a finite B-module. We are going to show more precisely that the Čech complex of $\mathcal{F}$ with respect to the covering $\mathfrak{V}$ is admissible and has cohomology groups which are finite Banach modules over B. By the usual arguments, we can assume that everything is strict. The proof of the finiteness theorem is rather functional-analytic: We are going to show that the Čech cohomology groups are cokernels of operators which are the difference between a surjective operator and one which is sufficiently close to having finite image. A version of a functional-analytic theorem due to Schwarz shows that such operators have finite cokernel. We will give a proof of Schwarz' theorem in Sections 3.4 and 3.5.

In Section 3.2, we are going to prove a version of the theorem on formal functions. Namely, for $b \in B$, we are going to establish an isomorphism

$$\varprojlim_i H^q(X, \mathcal{F}/b^i\mathcal{F}) \cong \varprojlim_i H^q(X, \mathcal{F})/b^i H^q(X, \mathcal{F}).$$

The proof of this fact is rather formal in nature and similar to the proof of the analogous theorem in algebraic geometry (which can be found in [Sta19, Sec. 02O7]).

In Section 3.3, we can finally show that the higher direct image sheaf $R^q\varphi_*(\mathcal{F})$ is associated to the finite B-module $H^q(X, \mathcal{F})$ and, hence, coherent. Since, in general, the q-th higher direct image sheaf is the sheaf associated to the presheaf $U \mapsto H^q(\varphi^{-1}(U), \mathcal{F})$, one only has to prove the isomorphism $H^q(\varphi^{-1}(U), \mathcal{F}) \cong B' \otimes_B H^q(X, \mathcal{F})$ whenever $U = \mathcal{M}(B') \subseteq Y = \mathcal{M}(B)$ is an affinoid subdomain. This can be reduced to the strict case, in which we will be able to prove the result making use of the formal function theorem.

3.1 The finiteness theorem

We shall be concerned with the proof of the following theorem.

Theorem 3.1.1. *Let $\varphi\colon X \to Y = \mathcal{M}(B)$ be a proper morphism of k-analytic spaces with affinoid base and let $\mathfrak{U} = (U_i)_{i\in I}$ and $\mathfrak{V} = (V_i)_{i\in I}$ be two finite affinoid coverings of X such that $V_i \Subset_Y U_i$ for all $i \in I$. If $\mathcal{F}$ is a coherent sheaf of $\mathcal{O}_X$-modules, then the Čech complex $C^\bullet(\mathfrak{V}, \mathcal{F})$ is admissible and its cohomology groups $H^q(\mathfrak{V}, \mathcal{F})$ are finite Banach B-modules.*

Corollary 3.1.2. *Let $\varphi\colon X \to Y = \mathcal{M}(B)$ be a proper morphism of k-analytic spaces with affinoid base and let $\mathcal{F}$ be a coherent sheaf of $\mathcal{O}_X$-modules. Then the cohomology groups $H^q(X, \mathcal{F})$ are finite B-modules.*

Proof. By Proposition 1.7.6, there exist two finite affinoid coverings $\mathfrak{U} = (U_i)_{i\in I}$, $\mathfrak{V} = (V_i)_{i\in I}$ such that $V_i \Subset_Y U_i$ for all $i \in I$. By Theorem 3.1.1, the Čech cohomology groups of $\mathcal{F}$ with respect to $\mathfrak{V}$ are finite Banach B-modules and, in particular, finite B-modules. Since X is separated, it follows from Proposition 2.6.2 that $H^q(X, \mathcal{F})$ coincides with $H^q(\mathfrak{V}, \mathcal{F})$ and therefore is finite as a B-module. □

In order to prove Theorem 3.1.1, let us first note that by Remark 2.6.3 and [Ber90, Prop. 2.1.11], we may assume that the field k is non-trivially valued, and that the spaces X and Y as well as the coverings $\mathfrak{U}$ and $\mathfrak{V}$ are strict.

Assume that k is non-trivially valued and let B be a Banach k-algebra. Let M and N be two Banach B-modules. We will write $B^\circ := \{ b \in B \mid |b| \leq 1 \}$, and similarly $M^\circ := \{ x \in M \mid |x| \leq 1 \}$. A bounded operator $f : M \to N$ is called *strictly completely continuous* if there is a sequence of bounded operators $f_i : M \to N$ such that $f = \lim_{i\to\infty} f_i$ in the operator norm of $\mathrm{Hom}(M, N)$ and such that there exists some $0 \neq c \in k^\circ$ such that for every $i \in \mathbb{N}$, the module $c \cdot f_i(M^\circ)$ is contained in some finite B°-submodule of N°. This condition should be viewed as designed to make the following theorem true:

Theorem 3.1.3 (Schwarz)**.** *Assume that the valuation on k is non-trivial, let B be a strictly k-affinoid algebra whose norm is the quotient norm with respect to some surjective homomorphism $\mathbb{T}_n \twoheadrightarrow B$. Let $f, g : M \to N$ be two bounded operators of Banach B-modules such that:*

(i) *The map f is surjective.*
(ii) *The map g is part of a sequence of bounded operators of Banach B-modules $L \xrightarrow{p} M \xrightarrow{g} N \xrightarrow{j} P$, where p is surjective, j is admissible and injective, and $j \circ g \circ p$ is strictly completely continuous.*

Then $(f - g)(M)$ is closed in N and the quotient $N/(f - g)(M)$ is a finite Banach B-module.

We postpone a proof of Theorem 3.1.3 to the end of this chapter. Our goal for now is to apply the theorem to two operators where N is th group of Čech cocycles $Z^q(\mathfrak{V}, \mathcal{F})$ and $(f - g)(M)$ is the group of Čech coboundaries $B^q(\mathfrak{V}, \mathcal{F})$, so that $N/(f - g)(M)$ will be the Čech cohomology group $H^q(\mathfrak{V}, \mathcal{F})$.

Our supply for strictly completely continuous maps is guaranteed by the following lemma:

Lemma 3.1.4. *Assume that the valuation on k is non-trivial, and let B be a strictly k-affinoid algebra. Let $\Phi : B\{T_1, \ldots, T_n\} \to A$, $T_i \mapsto f_i$ be a morphism of strictly k-affinoid algebras such that $\rho(f_i) < 1$ for all*

$i \in [1 \,..\, n]$. *Then Φ is a strictly completely continuous morphism of Banach B-modules.*

Proof. From $\rho(f_i) < 1$ it follows that the functions $f_i \in A$ are topologically nilpotent and hence that the family $(f^\nu)_{\nu \in \mathbb{N}^n}$ converges to 0. Now let $\Phi_\nu : B\{T_1, \dots, T_n\} = \hat{\bigoplus}_{\nu \in \mathbb{N}^n} B \cdot T^\nu \to A$ be the continuous operator which is given on $B \cdot T^\nu$ by $T^\nu \twoheadrightarrow f^\nu$ and vanishes elsewhere, so that Φ is the convergent sum of all the operators Φ_ν. Let us choose $c \in k^\circ$ in a way such that $|c||\Phi||f^\nu| \leq 1$ and $|c||f^\nu| \leq 1$ for all $\nu \in \mathbb{N}^n$ (this is possible, because $(f^\nu)_{\nu \in \mathbb{N}}$ is bounded). Then $c \cdot \Phi_\nu(B\{T_1, \dots, T_n\}) \subseteq A^\circ$ is generated by the element $c \cdot f^\nu$ as a B°-module. $\square$

Corollary 3.1.5. *Assume that the valuation on k is non-trivial, and let $\varphi : X \to Y = \mathcal{M}(B)$ be a morphism of strictly k-analytic spaces. Let $U, V \subseteq X$ be two strictly affinoid subdomains of X such that $V \Subset_Y U$. Finally, let $\mathcal{F}$ be a sheaf of coherent $\mathcal{O}_X$-modules. Then there exists a Banach B-module E and a surjective operator $p : E \to \mathcal{F}(U)$ such that the composition $E \to \mathcal{F}(U) \to \mathcal{F}(V)$ is a strictly completely continuous operator of Banach B-modules.*

Proof. We have $\mathcal{F}|_U \cong \widetilde{M}$ where M is some finite module over $A := \mathcal{O}_X(U)$, which carries a unique structure as a finite Banach A-module, and hence as a Banach B-module. We have $\mathcal{F}(V) \cong \mathcal{O}_X(V) \hat{\otimes}_A M$ as Banach B-modules. Since M is a finite Banach A-module, there is a surjective bounded operator $A^m \to M$ for some $m \in \mathbb{N}$. We obtain a commutative diagram

$$\begin{tikzcd} A^m \arrow[r] \arrow[d, two heads] & \mathcal{O}_X(V) \hat{\otimes}_A A^m \cong \mathcal{O}_X(V)^m \arrow[d, two heads] \\ M \arrow[r] & \mathcal{O}_X(V) \hat{\otimes}_A M. \end{tikzcd}$$

As one easily checks that post-composing a strictly completely continuous operator with a surjective one remains strictly completely continuous, we may assume that M is finite free. Since direct sums of strictly completely continuous operators are strictly completely continuous, we may even assume that $M = A$. But now the claim follows from Lemma 1.7.1 and Lemma 3.1.4. $\square$

Lemma 3.1.6. *Assume that the valuation on k is non-trivial, and let $\varphi\colon X \to Y = \mathcal{M}(B)$ be a proper morphism of strictly k-analytic spaces with affinoid base. Let $\mathfrak{U} = (U_i)_{i\in I}$ and $\mathfrak{V} = (V_i)_{i\in I}$ be two finite strictly affinoid coverings of X such that $V_i \Subset_Y U_i$ for all $i \in I$. Finally, let $\mathcal{F}$ be a sheaf of coherent $\mathcal{O}_X$-modules. Denote by*

$$r^q\colon C^q(\mathfrak{U}, \mathcal{F}) \to C^q(\mathfrak{V}, \mathcal{F})$$

the map induced by the restrictions along $V_i \subseteq U_i$. Then there exists a Banach B-module E^q and a surjective bounded operator $p^q\colon E^q \to C^q(\mathfrak{U}, \mathcal{F})$ such that $r^q \circ p^q$ is strictly completely continuous.

Proof. The map r^q is a direct sum of maps $\mathcal{F}(U) \to \mathcal{F}(V)$ where $U = U_{i_1} \cap \cdots \cap U_{i_q}$ and $V = V_{i_1} \cap \cdots \cap V_{i_q}$ for some indices $i_1, \dots, i_q \in I$ satisfy $U \Subset_Y V$ by Lemma 1.7.4 (iii). Hence the claim follows from Corollary 3.1.5. □

We are now ready to prove Theorem 3.1.1.

Proof of Theorem 3.1.1. We can choose a ground field extension $k \hookrightarrow k_r$ such that after scalar extension, the valuation of k is non-trivial and the spaces X and Y as well as the coverings $\mathfrak{U}$ and $\mathfrak{V}$ are strictly k-analytic. The Čech complex of the new sheaf $k_r \hat{\otimes}_k \mathcal{F}$ with respect to the covering $k_r \hat{\otimes}_k \mathfrak{V}$ is the completed tensor product of the old Čech complex with k_r, so by [Ber90, Prop. 2.1.11], we may assume that everything is strict. The claim is also clearly invariant under isomorphism of B as an affinoid k-algebra, so we may assume that the norm of B is given as the quotient norm inherited from some strict Tate algebra (so as to make Theorem 3.1.3 applicable).

From Banach's open mapping theorem, it follows that the admissibility of the coboundary operators is equivalent to having closed image. Hence we are done if we can show that the map

$$(d^{q-1}, 0)\colon C^{q-1}(\mathfrak{V}, \mathcal{F}) \oplus Z^q(\mathfrak{U}, \mathcal{F}) \to Z^q(\mathfrak{V}, \mathcal{F})$$

has closed image and finite cokernel. This map is the difference of the two operators

$$(d^{q-1}, r^q)\colon C^{q-1}(\mathfrak{V}, \mathcal{F}) \oplus Z^q(\mathfrak{U}, \mathcal{F}) \to Z^q(\mathfrak{V}, \mathcal{F})$$

and

$$(0, r^q) : C^{q-1}(\mathfrak{V}, \mathcal{F}) \oplus Z^q(\mathfrak{U}, \mathcal{F}) \to Z^q(\mathfrak{V}, \mathcal{F}).$$

Since by Proposition 2.6.2, the map $r^q : C^q(\mathfrak{U}, \mathcal{F}) \to C^q(\mathfrak{V}, \mathcal{F})$ induces isomorphisms on the cohomology groups, it follows that the first map (d^{q-1}, r^q) is surjective. Now let $p^q : E^q \to C^q(\mathfrak{U}, \mathcal{F})$ be as in Lemma 3.1.6. Writing E'^q for the inverse image of $Z^q(\mathfrak{U}, \mathcal{F})$ under p^q, the composition

$$C^{q-1}(\mathfrak{V}, \mathcal{F}) \oplus E'^q \xrightarrow{\mathrm{id} \times p^q} C^{q-1}(\mathfrak{V}, \mathcal{F}) \oplus Z^q(\mathfrak{U}, \mathcal{F}) \xrightarrow{(0, r^q)} Z^q(\mathfrak{V}, \mathcal{F}) \xrightarrow{\iota} C^q(\mathfrak{V}, \mathcal{F})$$

is strictly completely continuous. From Theorem 3.1.3 it follows now that $(d^q, 0) = (d^q, r^q) - (0, r^q)$ has closed image and that its cokernel $H^q(\mathfrak{V}, \mathcal{F})$ is a finite Banach B-module. $\square$

3.2 The theorem on formal functions

Let us fix a proper morphism $\varphi : X \to Y = \mathcal{M}(B)$ of k-analytic spaces with affinoid base throughout this section, as well as a coherent sheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules and an element $b \in B$. For each $i \in \mathbb{N}$, consider the exact sequence

$$\mathcal{F} \xrightarrow{b^i} \mathcal{F} \to \mathcal{F} / b^i \mathcal{F} \to 0,$$

where we write b both for the element of B and for its pullback to a function on X.

We obtain a complex

$$H^q(X, \mathcal{F}) \xrightarrow{b^i} H^q(X, \mathcal{F}) \to H^q(X, \mathcal{F} / b^i \mathcal{F}),$$

and hence a canonical morphism

$$\sigma^i : H^q(X, \mathcal{F}) / b^i H^q(X, \mathcal{F}) \to H^q(X, \mathcal{F} / b^i \mathcal{F}).$$

Taking inverse limits, we obtain a morphism

$$\sigma : \widehat{H^q(X, \mathcal{F})} = \varprojlim_i H^q(X, \mathcal{F}) / b^i H^q(X, \mathcal{F}) \to \varprojlim_i H^q(X, \mathcal{F} / b^i \mathcal{F})$$

of $\widehat{B}$-modules, where $\widehat{(-)}$ always denotes the b-adic completion. Our goal for this section is to prove the following theorem:

Theorem 3.2.1 (Theorem on formal functions)**.** *In the above situation, the morphism*

$$\sigma\colon \widehat{H^q(X,\mathcal{F})} \to \varprojlim_i H^q(X, \mathcal{F}/b^i\mathcal{F})$$

is an isomorphism of $\widehat{B}$-modules.

In order to simplify the notation, we write $H^q(\mathcal{G}) \coloneqq H^q(X, \mathcal{G})$ whenever $\mathcal{G}$ is a sheaf of $\mathcal{O}_X$-modules on X. Let us consider the direct sum $M^q(\mathcal{F}) \coloneqq \bigoplus_{i\in\mathbb{N}} H^q(b^i\mathcal{F})$ as a module over the blow-up algebra $S = \bigoplus_{i\in\mathbb{N}} S_i \coloneqq \bigoplus_{i\in\mathbb{N}} b^iB$, where the multiplication with the degree 1 element $b \in S_1 \subseteq S$ is given by the map $H^q(b^i\mathcal{F}) \to H^q(b^{i+1}\mathcal{F})$ which is induced from $\cdot b\colon\ b^i\mathcal{F} \to b^{i+1}\mathcal{F}$. Note that S is generated by $b \in S_1$ as an algebra over B and, in particular, a Noetherian ring.

Proposition 3.2.2. *The module $M^q(\mathcal{F})$ is a finitely generated module over the ring S.*

Proof. Let us first consider the case that $\mathcal{F}$ is b-torsion free. In this case, the map $\cdot b\colon\ b^i\mathcal{F} \to b^{i+1}\mathcal{F}$ is an isomorphism and so are the maps $H^q(b^i\mathcal{F}) \to H^q(b^{i+1}\mathcal{F})$. It follows that $M^q(\mathcal{F})$ is generated by its degree 0 part $H^q(\mathcal{F})$ as an S-module. Since by Corollary 3.1.2, $H^q(\mathcal{F})$ is finitely generated as a B-module, we are done in this case.

Next, we reduce the general case to the b-torsion free one. In order to do this, we consider the kernels of the maps $\cdot b^i\colon\ \mathcal{F} \to \mathcal{F}$. These form an increasing sequence of coherent subsheaves of $\mathcal{F}$, which becomes stationary at a certain subsheaf $\mathcal{G} \subseteq \mathcal{F}$ by Proposition 2.7.1. Then $\mathcal{F}/\mathcal{G}$ is b-torsion free and $b^{i_0}\mathcal{G} = 0$ for some $i_0 \in \mathbb{N}$. Let us now consider the b-filtration $\mathcal{G}_i \coloneqq \mathcal{G} \cap b^i\mathcal{F}$ of $\mathcal{G}$. By Proposition 2.7.2, this filtration is b-stable, i.e. there is an index $i_1 \in \mathbb{N}$ such that $\mathcal{G}_i = b^{i-i_1}\mathcal{G}_{i_1}$ for $i \geq i_1$. It follows that $\mathcal{G}_i = 0$ for $i \geq i_0 + i_1$. From Corollary 3.1.2, it follows that the S-module $N^q \coloneqq \bigoplus_{i\in\mathbb{N}} H^q(\mathcal{G}_i)$ is finitely generated over B and hence even more so over S.

The long exact cohomology sequence associated to the short exact sequence

$$0 \to \mathcal{G}_i \to b^i\mathcal{G} \to b^i(\mathcal{F}/\mathcal{G}) \to 0$$

yields an exact sequence

$$N^q \to M^q(\mathcal{F}) \to M^q(\mathcal{F}/\mathcal{G}).$$

As $\mathcal{F}/\mathcal{G}$ is b-torsion free, it follows that $M^q(\mathcal{F}/\mathcal{G})$ is a finite S-module. Since N^q is a finite S-module and S is a Noetherian ring, it follows that $M^q(\mathcal{F})$ is a finite S-module. □

Let us now consider the short exact sequence $0 \to b^i\mathcal{F} \to \mathcal{F} \to \mathcal{F}/b^i\mathcal{F} \to 0$ and denote by R_i the kernel of the induced map $H^q(\mathcal{F}) \to H^q(\mathcal{F}/b^i\mathcal{F})$, so that there is an exact sequence

$$0 \to R_i \to H^q(\mathcal{F}) \to H^q(\mathcal{F}/b^i\mathcal{F}).$$

Lemma 3.2.3. *The b-filtration $H^q(\mathcal{F}) \supseteq R_0 \supseteq R_1 \supseteq \ldots$ of the B-module $H^q(\mathcal{F})$ is b-stable, i.e. there is some $i_0 \in \mathbb{N}$ such that $R_i = b^{i-i_0}R_{i_0}$ for all $i \geq i_0$.*

Proof. By [AM69, Lem. 10.8], showing that the filtration is b-stable is equivalent to proving that the S-module $R^q(\mathcal{F}) := \bigoplus_{i\in\mathbb{N}} R_i$ is finitely generated. From the long exact cohomology sequence associated to $0 \to b^i\mathcal{F} \to \mathcal{F} \to \mathcal{F}/b^i\mathcal{F} \to 0$, it follows that $R_i = \mathrm{Im}(H^q(b^i\mathcal{F}) \to H^q(\mathcal{F}))$, and hence that $R^q(\mathcal{F})$ is a quotient of $M^q(\mathcal{F})$, which is finitely generated by Proposition 3.2.2. □

Consider the morphism $\sigma_i : H^q(\mathcal{F})/b^iH^q(\mathcal{F}) \to H^q(\mathcal{F}/b^i\mathcal{F})$ and set $K_i := \mathrm{Ker}(\sigma_i)$ and $Q_i := \mathrm{Coker}(\sigma_i)$.

Lemma 3.2.4. *The projective systems $(K_i)_{i\in\mathbb{N}}$ and $(Q_i)_{i\in\mathbb{N}}$ are null systems, i.e. for each $i \in \mathbb{N}$, there exists an index $j_0 \geq i$ such that for all $j \geq j_0$, the image of K_j in K_i vanishes, and similarly for the system $(Q_i)_{i\in\mathbb{N}}$.*

Proof. Let us first consider the system $(K_i)_{i\in\mathbb{N}}$. Note that $K_i = R_i/b^iH^q(\mathcal{F})$. By Lemma 3.2.3, there is some index i_0 such that $R_i = b^{i-i_0}R_{i_0}$ for $i \geq i_0$. Now given $i \in \mathbb{N}$, we can set $j_0 := i + i_0$ and see that the family $(K_i)_{i\in\mathbb{N}}$ is a null system.

In order to prove the claim about the system $(Q_i)_{i\in\mathbb{N}}$, we note first that from the long exact cohomology sequence associated to $0 \to b^i\mathcal{F} \to \mathcal{F} \to \mathcal{F}/b^i\mathcal{F} \to 0$, we can deduce that $Q_i = \mathrm{Im}(H^q(\mathcal{F}/b^i\mathcal{F}) \to H^{q+1}(b^i\mathcal{F}))$. Hence, $Q := \bigoplus_{i\in\mathbb{N}} Q_i$, as a submodule of $M^{q+1}(\mathcal{F})$ is a finitely generated S-module. From this description of Q_i as a quotient of $H^q(\mathcal{F}/b^i\mathcal{F})$, we can also read off that each Q_i is annihilated by some power of b. Since Q is a finitely generated S-module, there is some power b^r such that $b^rQ = 0$ if we view b^r as a degree 0 element $b^r \in S_0 \subseteq S$. Let us write b_1 for the element b viewed as a degree 1 element $b_1 \in S_1 \subseteq S$. Since all the Q_i are finitely generated B-modules, and Q is a finitely generated S-module, an argument as in the proof of [AM69, Lem. 10.8] shows that one can find integers $i_0 \in \mathbb{N}$ and $s \geq r$, such that $b_1^sQ_i = Q_{i+s}$ for all $i \geq i_0$. It follows that the canonical map $Q_{i+s} \to Q_i$ of the projective system $(Q_i)_{i\in\mathbb{N}}$ maps $Q_{i+s} = b_1^sQ_i$ to $b^sQ_i = 0$ for $i \geq i_0$, i.e. that $(Q_i)_{i\in\mathbb{N}}$ is a null system. □

Proof of Theorem 3.2.1. We split the exact sequence

$$0 \to K_i \to H^q(\mathcal{F})/b^iH^q(\mathcal{F}) \to H^q(\mathcal{F}/b^i\mathcal{F}) \to Q_i \to 0$$

into two short exact sequences

$$0 \to K_i \to H^q(\mathcal{F})/b^iH^q(\mathcal{F}) \to M_i \to 0$$

and

$$0 \to M_i \to H^q(\mathcal{F}/b^i\mathcal{F}) \to Q_i \to 0.$$

Since K_i is a null system and satisfies, in particular, the conditions of Mittag-Leffler, taking the inverse limit of the first short exact sequence shows that

$$\varprojlim_i H^q(\mathcal{F})/b^iH^q(\mathcal{F}) \xrightarrow{\sim} \varprojlim_i M_i.$$

Since the system $(M_i)_{i\in\mathbb{N}}$, as a quotient of the system $(H^q(\mathcal{F})/b^iH^q(\mathcal{F}))_{i\in\mathbb{N}}$ has surjective compatibility maps, it also satisfies the conditions of

Mittag-Leffler and taking the inverse limit of the second short exact sequence shows that

$$\varprojlim_i M_i \xrightarrow{\sim} \varprojlim_i H^q(\mathcal{F}\big/b^i\mathcal{F}),$$

because $(Q_i)_{i\in\mathbb{N}}$ is a null system. □

3.3 The proper mapping theorem

Theorem 3.3.1. *Let $\varphi\colon X \to Y = \mathcal{M}(B)$ be a proper morphism with affinoid base and let $Y' = \mathcal{M}(B') \subseteq Y = \mathcal{M}(B)$ be an affinoid subdomain. If $\mathcal{F}$ is a coherent sheaf of $\mathcal{O}_X$-modules on X, then the canonical map*

$$B' \otimes_B H^q(X,\mathcal{F}) \to H^q(X',\mathcal{F})$$

is an isomorphism of B'-modules, where we write $X' := \varphi^{-1}(Y')$.

The proof of this theorem can easily be reduced to a strict situation. In the strict situation, we will proceed by induction on the Krull dimension of B. The induction step will be based on the following observation:

Lemma 3.3.2. *Suppose that the valuation of k is non-trivial, let B be a strictly k-affinoid algebra of Krull dimension $d > 0$ and let $\mathfrak{m} \in$* MSpec(B) *be a maximal ideal. Then there exists an element $b \in \mathfrak{m}$ such that the Krull dimension of $B\big/b^iB$ is less than the Krull dimension of B for every $i \in \mathbb{N}$.*

Proof. By Noether normalization, there exists a finite embedding $\mathbb{T}_d \hookrightarrow B$ from a Tate algebra into B. Then $\mathfrak{n} := \mathfrak{m} \cap \mathbb{T}_d$ is a maximal ideal of $\mathbb{T}_d$, and since $d > 0$, $\mathfrak{n}$ cannot be the zero ideal. Hence, we find $0 \neq b \in \mathfrak{n}$. As $B\big/b^iB$ is finite over $\mathbb{T}_d\big/b^i\mathbb{T}_d$, it suffices to show that $\mathbb{T}_d\big/b^i\mathbb{T}_d$ has Krull dimension less than d. After a change of variables, we may assume that b, and hence b^i for all $i \in \mathbb{N}$, is a distinguished power series. By Weierstraß division, $\mathbb{T}_d\big/b^i\mathbb{T}_d$ is finite over $\mathbb{T}_{d-1}$, and we are done. □

The formal function theorem relates the (b-adic completion of the) cohomology of the sheaf $\mathcal{F}$ to the cohomology of the sheaves $\mathcal{F}/b^i\mathcal{F}$. By the following lemma, this is the same as the cohomology of the sheaf $\mathcal{F}_{b^i}$ on the space X_{b^i} which lies proper over $Y_{b^i} = \mathcal{M}(B/b^iB)$, so that by Lemma 3.3.2 we may hope to find an inductive proof of Theorem 3.3.1 over the Krull dimension of B.

Lemma 3.3.3. *Let $\varphi : X \to Y = \mathcal{M}(B)$ be a proper morphism and let $b \in B$. Consider the closed immersion $Y_b := \mathcal{M}(B/bB) \hookrightarrow Y = \mathcal{M}(B)$ and construct a cartesian square*

$$\begin{array}{ccc} X_b & \longrightarrow & Y_b \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y. \end{array}$$

Let $\mathcal{F}$ be a sheaf of coherent $\mathcal{O}_X$-modules on X and let $\mathcal{F}_b$ be its pullback to X_b. Then

$$H^q(X_b, \mathcal{F}_b) \cong H^q(X, \mathcal{F}/b\mathcal{F}).$$

Proof. Let $\mathfrak{U}$ be a finite affinoid covering of X and let $\mathfrak{U}_b$ be its pullback to X_b. Computing the cohomology of $\mathcal{F}/b\mathcal{F}$ in terms of the Čech complex associated to $\mathfrak{U}$, and the cohomology of $\mathcal{F}_b$ in terms of the Čech complex associated to $\mathfrak{U}_b$, and using the description of $\mathcal{F}_b$ on the members of $\mathfrak{U}_b$ offered by Remark 2.4.2, the claimed identity follows. □

Proof of Theorem 3.3.1. By Theorem 3.1.1, we can equip $H^q(X, \mathcal{F})$ and $H^q(X', \mathcal{F})$ with the structures of finite Banach modules over B, resp. B' and $B' \otimes_B H^q(X, \mathcal{F})$ coincides with the completed tensor product $B' \hat{\otimes}_B H^q(X, \mathcal{F})$. We can check that the map $B' \hat{\otimes}_B H^q(X, \mathcal{F}) \to H^q(X', \mathcal{F})$ is an isomorphism after performing a scalar extension to k_r, where $k \hookrightarrow k_r$ is a field extension such that all occurring spaces become strict. Since

$$k_r \hat{\otimes}_k (B' \hat{\otimes}_B H^q(X, \mathcal{F})) \cong (k_r \hat{\otimes}_k B') \hat{\otimes}_{k_r \hat{\otimes}_k B} H^q(k_r \hat{\otimes}_k X, k_r \hat{\otimes}_k \mathcal{F})$$

and

$$k_r \hat{\otimes}_k H^q(X', \mathcal{F}) \cong H^q(k_r \hat{\otimes}_k X', k_r \hat{\otimes}_k \mathcal{F})$$

by Theorem 3.1.1 and Remark 2.6.3, we may assume that everything is strict and that the valuation of k is non-trivial.

We are going to proceed by induction on the Krull dimension d of B. If $d = 0$, then B is an Artinian ring and hence a product of finitely many Artinian rings with a unique maximal ideal. If $B = B_1 \times B_2$, then $\mathcal{M}(B) = \mathcal{M}(B_1) \amalg \mathcal{M}(B_2)$ and it suffices to prove the result for B_1 and B_2 in place of B. Therefore, we may assume that B has a unique maximal ideal. In this case, the only strictly affinoid domains of $\mathcal{M}(B)$ are the empty set and $\mathcal{M}(B)$ itself, because the maximal spectrum of a strictly affinoid algebra lies dense in the Berkovich spectrum, so we are done.

Now suppose that $d > 0$ and that the result is already proven for all strictly affinoid algebras of dimension less than d. Then we have:

Claim 1. *If $b \in B$ is such that the dimension of $B\big/bB$ is less than d, then the result holds for the sheaf $\mathcal{F}\big/b\mathcal{F}$ in place of $\mathcal{F}$, i.e.*

$$B' \otimes_B H^q(X, \mathcal{F}\big/b\mathcal{F}) \cong H^q(X', \mathcal{F}\big/b\mathcal{F}).$$

In order to prove this claim, we consider the proper morphism $\varphi \colon X \to Y = \mathcal{M}(B)$, the affinoid subdomain $Y' = \mathcal{M}(B') \subseteq Y = \mathcal{M}(B)$ and the closed immersion $Y_b \coloneqq \mathcal{M}(B/bB) \hookrightarrow Y = \mathcal{M}(B)$ and construct the following commutative cube with cartesian faces:

$$\begin{tikzcd}
X'_b \arrow[rr] \arrow[dr] \arrow[dd] & & Y'_b \arrow[dr] \arrow[dd] & \\
& X' \arrow[rr] \arrow[dd] & & Y' \arrow[dd] \\
X_b \arrow[rr] \arrow[dr] & & Y_b \arrow[dr] & \\
& X \arrow[rr] & & Y
\end{tikzcd}$$

If we denote by $\mathcal{F}_b$ the pullback of $\mathcal{F}$ to X_b, then by induction hypothesis applied to the back face of the cube, we obtain

$$B'\big/bB' \otimes_{B/bB} H^q(X_b, \mathcal{F}_b) \cong H^q(X'_b, \mathcal{F}_b).$$

By Lemma 3.3.3, the left hand side can be identified with $B'/bB' \otimes_B H^q(X, \mathcal{F}/b\mathcal{F})$. Because b acts trivially on $H^q(X, \mathcal{F}/b\mathcal{F})$, this is the same as $B' \otimes_B H^q(X, \mathcal{F})$. Similarly, the right hand side can be identified with $H^q(X', \mathcal{F}/b\mathcal{F})$, which proves the claim.

Claim 2. *If $b \in B$ is such that the dimension of B/b^iB is less than the dimension of B for all $i \in \mathbb{N}$, then the canonical morphism*

$$\widehat{B'} \otimes_B H^q(X, \mathcal{F}) \to \widehat{B'} \otimes_{B'} H^q(X', \mathcal{F})$$

is an isomorphism of $\widehat{B'}$-modules, where $\widehat{B'}$ denotes the b-adic completion of B'.

Note that because $H^q(X, \mathcal{F}) \otimes_B B'$ and $H^q(X', \mathcal{F})$ are finite B'-modules, the tensor product with $\widehat{B'}$ is isomorphic to the b-adic completion. Hence, the map under consideration is the inverse limit of the maps

$$B'/b^iB' \otimes_B H^q(X, \mathcal{F}) \to B'/b^iB' \otimes_{B'} H^q(X, \mathcal{F}).$$

These maps fit into commutative diagrams

$$\begin{tikzcd}
B'/b^iB' \otimes_B H^q(X, \mathcal{F}) \arrow[r] \arrow[d] & B'/b^iB' \otimes_{B'} H^q(X', \mathcal{F}) \arrow[d] \\
B' \otimes_B H^q(X, \mathcal{F}/b^i\mathcal{F}) \arrow[r] & H^q(X', \mathcal{F}/b^i\mathcal{F}).
\end{tikzcd}$$

The lower maps are isomorphisms by Claim 1. The right maps become an isomorphism after passing to the inverse limit by Theorem 3.2.1. The left map is a tensor product of the map $H^q(X, \mathcal{F})/b^iH^q(X, \mathcal{F}) \to H^q(X, \mathcal{F}/b^i\mathcal{F})$ with the flat B-algebra B'. Therefore, its kernel, resp. cokernel is given by $B' \otimes_B K_i$, resp. $B' \otimes_B Q_i$, where we use the notation of Lemma 3.2.4. Since by the same lemma, the systems $(K_i)_{i \in \mathbb{N}}$ and $(Q_i)_{i \in \mathbb{N}}$ are null systems, the same is true for $(B' \otimes_B K_i)_{i \in \mathbb{N}}$, resp. $(B' \otimes_B Q_i)_{i \in \mathbb{N}}$. Hence, we can conclude, as in the proof of Theorem 3.2.1 that the left maps become isomorphisms after passing to the inverse limit. It follows that the same is true for the upper maps, which was the claim.

We are now ready to prove that the map $B' \otimes_B H^q(X, \mathcal{F}) \to H^q(X', \mathcal{F})$ is an isomorphism of B'-modules. It suffices to show that the map becomes an isomorphism after tensoring with $B'_{\mathfrak{m}'}$, where $\mathfrak{m}'$ runs through the maximal ideals of B'. Since the $\mathfrak{m}'$-adic completion $\widehat{B'_{\mathfrak{m}'}}$ is faithfully flat over $B'_{\mathfrak{m}'}$, it is enough to prove that the map

$$\widehat{B'_{\mathfrak{m}'}} \otimes_B H^q(X, \mathcal{F}) \to \widehat{B'_{\mathfrak{m}'}} \otimes_{B'} H^q(X', \mathcal{F})$$

is an isomorphism. Let $\mathfrak{m}$ be the unique maximal ideal of B such that $\mathfrak{m}' = \mathfrak{m}B$ and choose b as in Lemma 3.3.2. Then, because b, considered as an element of B', is an element of $\mathfrak{m}'$, the $\mathfrak{m}'$-adic completion of B' factors over the b-adic completion of B'. Therefore, it suffices to show that the map

$$\widehat{B'} \otimes_B H^q(X, \mathcal{F}) \to \widehat{B'} \otimes_{B'} H^q(X', \mathcal{F})$$

is an isomorphism, where $\widehat{B'}$ denotes the b-adic completion. This is precisely Claim 2 above, so we are done. □

Theorem 3.3.4. *Let $\varphi\colon X \to Y = \mathcal{M}(B)$ be a proper morphism of k-analytic spaces with affinoid base and let $\mathcal{F}$ be a coherent sheaf of $\mathcal{O}_X$-modules. Then the q-th higher direct image sheaf $R^q\varphi_*(\mathcal{F})$ is associated to the finite B-module $H^q(X, \mathcal{F})$.*

Proof. We consider the composition of morphisms of G-topological spaces

$(X, \text{analytic G-topology}) \to (Y, \text{analytic G-topology}) \to (Y, \text{weak G-topology}).$

Since the second map induces an equivalence on the respective categories of sheaves, we may view Y as equipped with the weak G-topology. By [Sta19, Lem. 072W], the higher direct image sheaf $R^q\varphi_*(\mathcal{F})$ is the sheaf associated to the presheaf

$$V \mapsto H^q(\varphi^{-1}(V), \mathcal{F})$$

on Y. However, in Theorem 3.3.1 we have seen that this presheaf is already a sheaf, namely the sheaf associated to $H^q(X, \mathcal{F})$. □

Corollary 3.3.5. *Let $\varphi\colon X \to Y$ be a proper morphism of k-analytic spaces and let $\mathcal{F}$ be a sheaf of coherent $\mathcal{O}_X$-modules on X. Then the higher direct image sheaves $R^q\varphi_*(\mathcal{F})$ are again coherent.*

Proof. This can be checked locally on an affinoid covering of Y, so it follows from Theorem 3.3.4. $\square$

3.4 Almost Noetherian modules (*)

Let k be a complete, non-trivially valued, non-archimedean field with valuation ring $k^\circ = \{ a \in k \mid |a| \le 1 \}$ and let R be a k°-algebra. An R-module E is called *α-almost finitely generated* for a real number $0 < \alpha < 1$ if there is a finitely generated R-submodule $E' \subseteq E$ such that $a \cdot E \subseteq E'$ for all $a \in k^\circ$ satisfying $|a| \le \alpha$. The module is called *almost finitely generated* if it is α-almost finitely generated for all $0 < \alpha < 1$.

We call an R-module E *almost Noetherian* if all its submodules are almost finitely generated. The algebra R is *almost Noetherian* if it is almost Noetherian as a module over itself.

Remark 3.4.1. Obviously, every finitely generated module is almost finitely generated and consequently, every Noetherian module is almost Noetherian. In particular, every Noetherian algebra is almost Noetherian. As is well-known, the valuation ring k° is Noetherian if and only if the valuation group of k is a discrete subgroup of $\mathbb{R}_{>0}$. In this case, it is shown in [DG71, Chap. 0, Prop. 7.5.2] that, more generally, the Tate algebras $k^\circ\{T_1, \dots, T_n\}$ with coefficients in k° are Noetherian. Without any requirements on the value group, it is still true that k° and, more generally, $k^\circ\{T_1, \dots, T_n\}$ are almost Noetherian, which we want to prove in the following.

From now on, up to Corollary 3.4.8, we are going to assume that k has dense value group in $\mathbb{R}_{>0}$.

Lemma 3.4.2. *The valuation ring k° is almost Noetherian.*

Proof. Let $\mathfrak{a} \subseteq k^\circ$ be an ideal and let $0 < \alpha < 1$. We set $s := \sup_{c \in \mathfrak{a}} |c|$. Now choose $b \in k^\circ$ such that $\alpha s \le |b| < s$ and set $\mathfrak{a}' := \langle b \rangle \subseteq k^\circ$. Then $\mathfrak{a}' \subseteq \mathfrak{a}$ and for $a \in k^\circ$ with $|a| \le \alpha$, we have $a\mathfrak{a} \subseteq \mathfrak{a}'$. This shows that every ideal of k° is almost finitely generated. $\square$

Lemma 3.4.3. *Let R be a k°-algebra and let $0 \to E_1 \xrightarrow{\iota} E \xrightarrow{\pi} E_2 \to 0$ be a short exact sequence of R-modules.*

(i) *If E is α-almost finitely generated, then so is E_2.*

(ii) *If $0 < \sqrt{\alpha} < \gamma < 1$ and if E_1 and E_2 are γ-almost finitely generated, then E is α-almost finitely generated.*

Proof. The first statement is clear, so let us proceed with the second one. By assumption, there exist finitely generated submodules $E_1' \subseteq E_1$ and $E_2' \subseteq E_2$ satisfying $cE_1 \subseteq E_1'$ and $cE_2 \subseteq E_2'$ for all $c \in k^\circ$ with $|c| \leq \gamma$. Choosing preimages of a finite generating system of E_2', we obtain a finitely generated submodule $\tilde{E_2'}$ of E satisfying $\pi(\tilde{E_2'}) = E_2'$. Now we set $E' := \iota(E_1') + \tilde{E_2'}$ and claim that $aE \subseteq E'$ for all $a \in k^\circ$ with $|a| \leq \alpha$, which would prove the lemma.

As the valuation on k is non-discrete, we find a $c \in k^\circ$ satisfying $\sqrt{|a|} \leq |c| \leq \gamma$. Because k° is a valuation ring, we then have $c^2 \mid a$. By choice of E_2', we have $\pi(cE) = cE_2 \subseteq E_2'$, which implies $cE \subseteq \iota(E_1) + \tilde{E_2'}$. Finally, we find

$$aE \subseteq c^2E \subseteq c(\iota(E_1) + \tilde{E_2'}) \subseteq \iota(E_1') + \tilde{E_2'} = E'. \qquad \square$$

Corollary 3.4.4. *Let R be a k°-algebra and let $0 \to E_1 \to E \to E_2 \to 0$ be a short exact sequence of R-modules. If E_1 and E_2 are almost finitely generated, then the same is true for E.*

Proof. Let $0 < \alpha < 1$ be given. Then we can choose γ such that $0 < \sqrt{\alpha} < \gamma < 1$. By assumption, E_1 and E_2 are γ-almost finitely generated. By Lemma 3.4.3, E is α-almost finitely generated. $\square$

Corollary 3.4.5. *Let R be a k°-algebra and let $0 \to E_1 \to E \to E_2 \to 0$ be a short exact sequence of R-modules. Then E is almost Noetherian if and only if E_1 and E_2 are almost Noetherian.*

Corollary 3.4.6. *If R is an almost Noetherian k°-algebra, then every finite R-module is almost Noetherian.*

Theorem 3.4.7. *Let $R = k^\circ\{T_1, \dots, T_n\}$ be a Tate algebra with coefficients in k°. Then every finite R-module E is almost Noetherian.*

Proof. By Corollary 3.4.6, we can reduce to the case $E = R$. Now we proceed by induction on n. The case $n = 0$ was handled in Lemma 3.4.2. Hence, we assume that $n > 0$ and that the result has already been shown for $n - 1$.

Let $0 \neq \mathfrak{a} \subseteq R$ be an ideal and $0 < \alpha < 1$. We want to show that $\mathfrak{a}$ is α-almost finitely generated. First we choose γ such that $\sqrt{\alpha} < \gamma < 1$. Then we set $\beta := \sup_{h \in \mathfrak{a}} |h|$ and choose $g \in \mathfrak{a}$ with $|g| > \gamma\beta$. Let $c \in k^\circ$ be a coefficient of maximal absolute value of g, so that $|c| = |g|$. Then the element $f = c^{-1}g \in k^\circ$ has Gauß norm 1. After a norm-preserving change of variables, we can assume that f is T_n-distinguished. By Weierstraß division, $R/\langle f \rangle$ is finite over $k^\circ\{T_1, \dots, T_{n-1}\}$. By induction hypothesis, $R/\langle f \rangle$ is almost Noetherian over $k^\circ\{T_1, \dots, T_{n-1}\}$, and hence over R. Now consider the short exact sequence

$$0 \to \langle f \rangle \to R \xrightarrow{\pi} R/\langle f \rangle \to 0$$

and the induced short exact sequence

$$0 \to \langle f \rangle \cap \mathfrak{a} \to \mathfrak{a} \to \pi(\mathfrak{a}) \to 0.$$

Since $\pi(\mathfrak{a}) \subseteq R/\langle f \rangle$ is almost finitely generated, it suffices to show that $\langle f \rangle \cap \mathfrak{a}$ is γ-almost finitely generated. In order to do this, we set $E_1' := \langle g \rangle \subseteq \langle f \rangle \cap \mathfrak{a}$. We have to show that for $a \in k^\circ$ with $|a| \leq \gamma$, we always have $a \cdot (\langle f \rangle \cap \mathfrak{a}) \subseteq \langle g \rangle$. Since f and g only differ by an element of k°, it suffices to check that $|h| \leq |g|$ for all $h \in a \cdot (\langle f \rangle \cap \mathfrak{a}) \subseteq a\mathfrak{a}$. This follows from the choice of g. □

Corollary 3.4.8. *Let k be a non-trivially valued field (discretely or non-discretely valued) and let B be a strictly k-affinoid algebra equipped with the residue norm inherited from a Tate algebra $k\{T_1, \dots, T_n\}$. Then $B^\circ := \{ b \in B \mid |b| \leq 1 \}$ is an almost Noetherian k°-algebra.*

Proof. Suppose that the norm on B is given by a surjective homomorphism $\Phi\colon k\{T_1, \dots, T_n\} \twoheadrightarrow B$. Then, as the kernel of Φ is strictly closed by [BGR84, Cor. 5.2.7/8], Φ restricts to a surjective homomorphism $k^\circ\{T_1, \dots, T_n\} \twoheadrightarrow B^\circ$. Hence, the result follows from Theorem 3.4.7 (and Remark 3.4.1). □

3.5 The theorem of Schwarz (*)

Let k be a non-trivially valued non-archimedean field and let B be a k-affinoid algebra. Let M, N be two Banach B-modules. A bounded operator $f : M \to N$ has *finite image* if $f(M)$ is a finitely generated B-module. It has *c-strictly finite image* for some $0 \neq c \in k^\circ$ if $c \cdot T(M^\circ)$ is contained in a finite B°-submodule of N°. Using the Noetherianity of B, one checks easily that the sets of operators with c-strictly finite image, where c runs through $k^\circ \setminus \{0\}$, ordered by divisibility, form a directed family of subspaces of the space of all finite image operators.

The closure of the space of all operators with finite image in $\mathrm{Hom}(M, N)$ is called the space of *completely continuous* operators. The closure of the space of operators with c-strictly finite image is called the space of *c-strictly completely continuous* operators. An operator $f : M \to N$ is called *strictly completely continuous* if it is c-strictly completely continuous for some $0 \neq c \in k^\circ$.

Lemma 3.5.1. *Let M, N be two Banach k-vector spaces. Then the set of surjective bounded operators $f : M \to N$ is open in* $\mathrm{Hom}(M, N)$ *with the operator norm.*

Proof. Let $f : M \to N$ be surjective. By Banach's open mapping theorem, f is admissible, so there exists an $\alpha \in \mathbb{R}_{>0}$ such that for every $y \in N$, there exists an $x \in M$ such that $f(x) = y$ and $|x| \leq \alpha|y|$. We claim that whenever $g \in \mathrm{Hom}(M, N)$ satisfies $|g| < \alpha^{-1}$, then $f - g$ is surjective as well.

Indeed, let $y_0 \in N$ be arbitrary. Then we find $x_0 \in M$ such that $f(x_0) = y_0$ and $|x_0| \leq \alpha|y_0|$. Then $y_1 := g(x_0)$ satisfies $|y_1| \leq (|g|\alpha)|y_0|$. Inductively, we construct x_n as a preimage of y_n such that $|x_n| \leq \alpha|y_n| \leq \alpha(|g|\alpha)^n|y_0|$ and set $y_{n+1} := g(x_n)$, so that $|y_{n+1}| \leq (|g|\alpha)^{n+1}|y_0|$ holds. The series $x := \sum x_i$ converges and satisfies $(f - g)(x_0) = y_0$. This shows the surjectivity of $f - g$. □

Theorem 3.5.2. *Let B be a k-affinoid algebra and let $f, g : M \to N$ be two bounded operators, where f is surjective and g is completely continuous. Then $(f - g)(M)$ is closed in N and $N/(f - g)(M)$ is a finite Banach B-module.*

Proof. By the previous lemma, small perturbations of f remain surjective. Since g can be approximated with arbitrary precision by finite image operators, we may assume that g has finite image. We note that $f(\mathrm{Ker}(g)) \subseteq (f-g)(M)$, so that we have a commutative diagram of surjective homomorphisms of B-modules

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[d] & N \arrow[d] \\ M\big/\mathrm{Ker}(g) \arrow[r] & N\big/(f-g)(M). \end{tikzcd}$$

It follows that $N\big/(f-g)(M)$ is a finite B-module. If we equip it with a norm making it a finite Banach B-module (which is, a priori, not necessarily equivalent to the quotient norm inherited from N), then the map $M \to M\big/\mathrm{Ker}(g) \to N\big/(f-g)(M)$ is bounded, because maps between finite Banach B-modules are bounded. Since f is surjective, it follows from Banach's open mapping theorem that $N \cong M\big/\mathrm{Ker}(f)$ as Banach B-modules. The universal property of the quotient $M\big/\mathrm{Ker}(f)$ implies that the right map in the diagram is bounded and hence that $(f-g)(M) = \mathrm{Ker}(N \to N\big/(f-g)(M))$ is closed in N. □

Lemma 3.5.3. *Let B be a strictly k-affinoid algebra whose norm is given by the quotient norm inherited from a Tate algebra $\mathbb{T}_n$ and let $M \xrightarrow{g} N \xrightarrow{j} P$ be bounded operators of Banach B-modules, where j is admissible and injective. Assume that M is topologically free, i.e. isomorphic to a completed direct sum $\hat{\bigoplus}_{\lambda\in\Lambda} B$ as a Banach B-module. If $j \circ g$ is strictly completely continuous, then the same is true for g.*

Proof. We may assume that M is equal to $\hat{\bigoplus}_{\lambda\in\Lambda} B$. Replacing N by its image in P, we may assume that j is an isometric embedding. By rescaling the norm on N and P, we may also assume that g is contractive. The map $j \circ g$ is c-strictly completely continuous for some $c \in k^\circ \setminus \{0\}$. By rescaling the norms on N and P further, we may assume that $c = 1$, i.e. that $j \circ g$ is a limit of a sequence of operators $h : M \to P$ such that $h(M^\circ)$ is contained in a finite B°-submodule of P°.

Now let us fix some $a \in k^\circ$ with $0 < |a| < 1$. I claim that g is a-strictly completely continuous. To prove this, it suffices to construct, for every $\varepsilon > 0$, a bounded operator $g' : M \to N$ such that

(i) $|g - g'| \leq \varepsilon/|a|$.

(ii) g' has a-strictly finite image, i.e. $ag'(M^\circ)$ is contained in a finite B°-submodule of N°.

We choose $h \colon M \to P$ such that $h(M^\circ)$ is contained in a finite B°-submodule of P° and such that $|j \circ g - h| \leq \varepsilon$. From Corollary 3.4.8 it follows that $h(M^\circ)$ is almost finitely generated, so there exists a finitely generated B°-submodule $E' \subseteq h(M^\circ)$ such that $a \cdot h(M^\circ) \subseteq E'$. Let us fix generators $y_1, \dots, y_r$ of E' and let $x_1, \dots, x_r \in M^\circ$ be such that $h(x_i) = y_i$. We set $z_i \coloneqq g(x_i) \in N^\circ$. Then $|y_i - j(z_i)| \leq \varepsilon$ for all $i \in [1 \,..\, r]$. From $ah(M^\circ) \subseteq E'$ it follows that there are elements $b_{i\lambda} \in B$ satisfying $|b_{i\lambda}| \leq 1/|a|$ such that

$$h(e_\lambda) = \sum_{i=1}^{r} b_{i\lambda} y_i.$$

We define $g' : M \to N$ by stipulating $g'(e_\lambda) = \sum_{i=1}^{r} b_{j\lambda} z_j$. Then we have

$$\begin{aligned} |g(e_\lambda) - g'(e_\lambda)| &\leq \max\{\, |j(g(e_\lambda)) - h(e_\lambda)|, |h(e_\lambda), j(g(e_\lambda))| \,\} \\ &\leq \max\{\, \varepsilon, \varepsilon/|a| \,\} \\ &= \varepsilon/|a| \end{aligned}$$

and hence $|g' - g| \leq \varepsilon/|a|$. Since $ag'(M^\circ) \subseteq \langle\, z_1, \dots, z_r \,\rangle \subseteq N^\circ$ by construction of g', we are done. □

Theorem 3.5.4. *Let B be a strictly k-affinoid algebra whose norm is the quotient norm inherited from some Tate algebra $\mathbb{T}_n$. Let $f, g : M \to N$ be two bounded operators of Banach B-modules such that:*

(i) *The map f is surjective.*

(ii) *The map g is part of a sequence of bounded operators of Banach B-modules $L \xrightarrow{p} M \xrightarrow{g} N \xrightarrow{j} P$, where p is surjective, j is admissible and injective, and $j \circ g \circ p$ is strictly completely continuous.*

Then $(f - g)(M)$ is closed in N and the quotient $N\big/(f - g)(M)$ is a finite Banach B-module.

Proof. Choosing a surjective operator $p' : L' \twoheadrightarrow L$ from a topologically free Banach module L' onto L and replacing p by $p \circ p'$, we may assume that L is topologically free. Then it follows from Lemma 3.5.3 that $g \circ p$ is strictly completely continuous and in particular completely continuous. Replacing f by $f \circ p$ and g by $g \circ p$, we may assume that g is completely continuous. But then the statement follows from Theorem 3.5.2. $\square$

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